MATHEMATICAL EMBODIMENT AND UNDERSTANDING

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This paper considers how reflection on practical activities can lead to meaningful embodiment of mathematical concepts in a way that integrates practice and theory. In particular, it focuses on how an ***action*** *such as translating an object on a table can be conceptualised as a mathematical* ***concept****—in this case a free vector—by shifting the focus of attention away from the actions themselves to the* ***effect*** *of the actions. The practical theory that arises has wide applications in mathematics involving processes that are symbolised and then conceived as mathematical concepts.*

'*I hear, and I forget; I see, and I remember; I do, and I understand.*'

saying of Confucius, quoted in Nuffield (1967)

'*I do and I understand how, I reflect and I understand why.*' Anna Poynter (2004)

As a teacher of mathematics for many years I became concerned how students seemed to be able to learn to do techniques to score highly on examinations, yet seemed not to be able to apply their knowledge in slightly different situations, or to retain their skills for ready use in subsequent courses. After talking with students and teachers, I realised that the accepted mode of teaching that builds from a variety of different practical contexts might be part of the problem. I hypothesised:

> "If only students could concentrate on the simplicity of the mathematical idea instead of the many complications connected to different contexts, then they should be in a better position to solve problems in novel situations."

The focus of this research is on how to encourage students to reflect on the mathematical ideas in a specific context to give meaning to the symbolism that arises in the mathematical theory. Skemp (1971) suggests that the mathematical idea should be built by focusing on one particular context that has the least 'noise' (distracting elements) to develop the relevant mathematical concepts in a way that can then be applied to other contexts. The approach developed in this study focuses on the case of vector, but it is widely applicable to all areas where physical actions are translated into mathematical symbolism. It follows on from a preliminary study (Watson[1] 2002), which describes the background leading to the main study outlined here.

In constructing the mathematical concept of vector, the science education literature shows ample evidence of a range of 'false intuitions' that may arise (Aguirre & Erickson, 1984, Jagger, 1988, Graham & Berry, 1997) using vectors as forces and journeys. In my approach I chose to start with the idea of physical transformations of

[1] Anna Poynter published earlier materials in the name of Anna Watson before her recent marriage.

an object on a flat plane, as used in the mathematics text-book, Pledger *et al.* (1996). This approach was based both on practical experience and on analysis of theories of cognitive development.

One simple practical observation led to a significant theoretical advance which linked the embodied theory of Lakoff & Núñez (2002) and theories concerning the way that mathematical procedures are interiorized as processes and symbolised as mathematical concepts in arithmetic and algebra (for example, Gray & Tall, 1994).

A student whom I shall call 'Joshua' explained that different actions can have the same '*effect*' (Watson, 2002). For example, he saw the combination of one translation followed by another as having the 'same effect' as the single translation corresponding to the sum of the two vectors. He also observed that solving problems with velocities or accelerations is mathematically the same: "the only difference is that one is metres per second and the other metres per second squared." He was able to operate with a free vector as a flexible mental concept in different contexts.

This single observation linked practical activities found in many classrooms to a theoretical framework of constructing mathematical ideas. By a 'delicate shift of attention' (Mason, 1989) one could change one's focus from the individual steps of an activity to the overall *effect* of that activity. This change of focus in the embodied world of physical action and mental reflection corresponds to the theoretical shift of interiorizing a sequence of actions as an overall process, and then encapsulating that process as a mental object. A free vector is such a concept, involving a process of translation and a concept of free vector (as an arrow of given magnitude and direction or symbolically as a column vector). However, once the idea of free vector is constructed and, more importantly, of the *sum* of free vectors is conceived using the equivalent triangle or parallelogram laws, then this concept is available to be applied in many different contexts, including, including vectors as journeys and as forces.

To encourage students to construct the concept of free vector for themselves, consistent with the mathematical theory of vectors, the lessons began with physical activities in which students performed the action of translating a triangle on a table. The triangle functioned as a 'base object' on which the translations acted and, by focusing on the *effect* of the translation, students could gain experience that any arrow of a given magnitude and direction could be used to represent a translation of that magnitude and direction. The concept of an arrow as a free vector was then made the focus of attention and 'free vectors' added by moving them 'nose to tail', to relate to the single vector that 'has the same effect'. The activities also looked at different ways in which the vectors could be added (for example using the triangle or parallelogram methods) to see their equivalence.

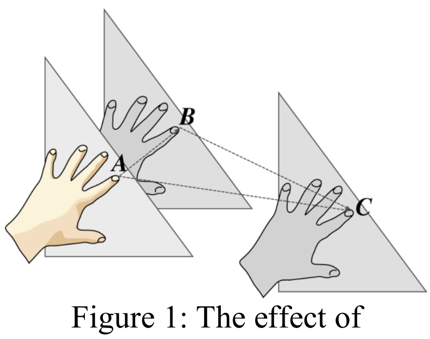


Figure 1: The effect of successive translations

The students' own construction of the notion of free vector was supported by activities and discussions in reflective plenary sessions. Building on the theory of Barbara Jaworski (1993) who proposed that, after activities in which students participate, the teacher should create the situation in which (s)he can enable them to construct meaningful concepts. This proved advantageous as it linked with the idea of using plenaries in the English National Curriculum. In these plenaries, students were encouraged to build a meaningful concept of free vector as an encapsulated object that they could operate on in different contexts, mathematical as well as physical.

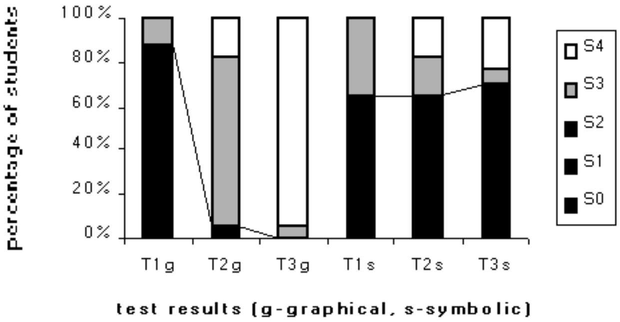
The methodology of the main study (Poynter 2004) involved comparing two classes in the same school: Group A taught by the researcher using an embodied approach focusing on the effect of a translation, Group B taught using the standard text-book approach by a comparable teacher in parallel. The changes were monitored by a pre-test, post-test and delayed post-test and a spectrum of students were selected for individual interviews. The test was shown to colleagues in mathematics and physics to determine what aspects of the test the teachers thought the students might find difficult. To assess the development a sequence of stages were assigned to both embodied geometric responses and symbolic responses broadly following the fundamental cycle of concept construction, first with the vector concept in one dimension, followed by the compression of journey to free vector in two dimensions:

Stage	Graphical	Symbolic
0	No response	No response
1	Journey in one dimension	A signed number
2	Arrow as a journey from *A* to *B*	Horizontal and vertical components
3	Shifts with same magnitude and direction	Column vector as relative shift
4	Free vector	Vector **u** as a manipulable symbol

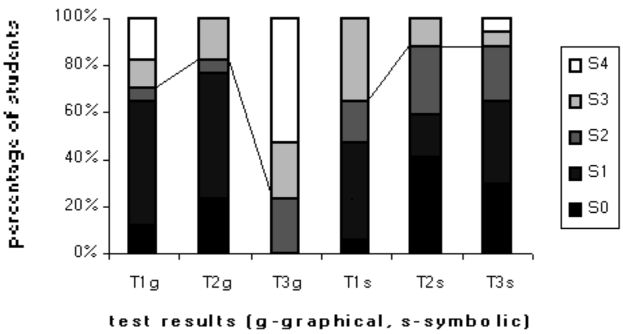
Figure 2: Fundamental cycle of concept construction of free vector addition

A problem-sheet of vector problems was given to the students on three occasions (pre-test T1, post-test T2 and delayed post-test T3). The questions mainly related to graphical properties of vectors and their addition with applications to both vectors as journeys and as forces. For example some questions asked students to add vectors that were placed in unusual positions, such as two arrows meeting at a point or two arrows crossing each other. These 'singular' questions which did not fit the prototypical images of forces acting from a single point, or of journeys following one after another, were likely to cause difficulties for students who had not constructed a flexible concept of 'free vector'.

The results showing the students' development through four stages of vector addition are represented in figures 1 and 2. T1g, T2g and T3g represent the graphical results, and T1s, T2s and T3s the symbolic results of each group in the three tests.



(a) Group A



(b) Group B

Table 1: Comparative General Development of Groups A and B

The stages in each column are represented vertically in successive shades from stage 0 (black) at the bottom to stage 4 (white) at the top. The lines joining successive columns show the changing levels of the point between the two higher stages (3 and 4) and the three lower stages (0, 1, 2). In the test, most of the questions were of a graphical nature but several invited alternative responses (which might be given in terms of adding components), hence it is natural that the major changes occurred in the increasing sophistication of graphical development.

In graphical development, Group A started *below* Group B, and ended *above*. Those in the top two stages (3 and 4) increase in group A from 12% to 94% from pre-test to delayed post-test while those in Group B increase from 65% to 76%. The t-test for graphical changes between pre-test T1 and delayed post-test T3 were highly significant for Group A ($t=3.83$ at $p<0.01$) and not significant for Group B.

The numerical responses that occurred at the two higher levels decrease slightly in both groups, Group A from 35% to 29%, Group B from 35% to 12%. The overall changes for the symbolic mode were not significant for either group.

The results support the hypothesis that

> Students who can concentrate on the *effect* of actions rather than actions themselves are more likely to build the concept of free vector as a flexible concept, which can be used by students after a longer period of time and not only just after the experiment.

In addition to this overall quantitative analysis extracted from a more in-depth analysis (Poynter, 2004), interviews were held with selected students and staff to triangulate qualitative and quantitative data. While the teachers were aware of the kind of problems the students might encounter, there was a distinct difference between the perceptions of the physics teachers, who related vector ideas to specific contexts involving forces or journeys, and the mathematics teachers, who were implicitly aware of the progression in building the concept of free vector given in figure 2 without being explicitly aware of the cognitive theory (Poynter & Tall, 2005a). Further detail relating theory and practice is in Poynter & Tall, (2005b).

Interviews with students revealed a difference in the language that students used when operating at different levels of cognitive development in figure 2. Students operating at lower cognitive levels use procedures without reference to the concept of free vector, while students operating at higher cognitive levels spoke of the concept of vector as a mental entity that operated in the same way in different contexts.

For example, one of the high attaining students from Group A, when asked how he tackled the singular case in which two vectors met at one point, said: "I was sliding the vectors so one is at the end of the other, so that they are nose to tail, and then drew a resultant. [...] I worked out the length and direction, did them in the **i** and **j** direction and added them together." When asked how he answered questions set in different contexts (forces and displacement), he answered: "They are the same. You could do them in the **i** and **j** direction and add the together, or you could draw them so they are nose to tail and draw the resultant." When asked how he approached the singular question with two vectors meeting at a point, he responded: "I didn't know how to do this and this was like a second thought [...] I was making my own way of doing it." He went on to correctly solve the problem. His answers show that he has built a cognitive unit, which he was confident to apply to an unfamiliar situation. He also implied that every time he looked at the *result* of the addition, which meant that he concentrated on the *effect* of the addition, in both symbolic and graphical mode.

On the other hand another student, this time from Group B, put two vectors together but did not draw the resultant. When asked how he went about answering the question he responded: "I did not know what you meant by 'add the two vectors', so I assumed it was put them together as arrows." When asked to explain a bit more what he understood by addition, he answered: "I understand the addition as showing the total movement." Evidently his task ended when he completed the action of movement along the path and the further step of drawing an equivalent path with the same effect was not conceived as part of the task. When asked how he tackled two questions, one asking him to draw and add three forces and another to draw and add two displacements, he answered: "Apart from the fact that there is an extra force in the first one, they are exactly the same." However, he concentrated in both situations as if they applied to forces and answered them in that context, not as free vectors in a mathematical context. When asked if he noticed the contexts are different he said: "Ah […] the forces are not necessarily vectors; I don't think they are movements." His experience had already shown him that forces acting on an object do not have to have to cause a movement, providing him with separate experiences for the two concepts. He kept the notions of force and journey as separate concepts and did not see them as being related to a common notion of vector. It was also apparent that he had not developed an awareness of some of the more subtle mathematical properties of vectors. For instance, there was no indication in his responses that he knew that the addition of vectors is commutative.

WIDER THEORETICAL IMPLICATIONS

The notion of 'effect' of actions on base objects has applications in many areas of mathematics where the construction of mathematical concepts is encapsulated from processes. In essence, the idea of two actions having the same effect translates into the notion of two mathematical concepts being equivalent. For instance, equivalent fractions are different sharing procedures with the same effect, equivalent algebraic expressions are different procedures of evaluation with the same effect, and so on. A major line of research is to investigate the use of the focus on 'effect' in giving cognitive meaning to such mathematical concepts.

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