Chapter 3

Towards the Theoretical Framework

3.1 Introduction

This chapter develops the theoretical framework for the empirical research which follows. It begins by describing the nature of the curriculum in the UK and focuses on the experiences that the students in the study will have encountered previously in mathematics and physics.

Early ideas that lead to the notion of vector relate to physical experiences of forces in a single dimension, involving the combination of forces in different directions. In physics this leads to the study of forces resolved in horizontal and vertical components and combining forces by adding their components in each direction. The introduction of vectors in mathematics passes through a sequence in which transformations are re-conceptualised as vectors, which follows a sequence similar to process-object encapsulation. This in turn leads to the possibility of a theoretical framework, which studies how the students cope with successive stages of process-object encapsulation, both graphically as arrows with magnitude and direction and numerically as separate horizontal and vertical components.

However, before embarking on the development of such a framework, we consider relevant research in science education with respect to students' understanding of vectors in Physics and Mechanics. This will lay the groundwork for a preliminary study described in chapter 4 in which the theoretical framework will be further refined before the design of the main studies that follow.

3.2 The school based situation

In the English system, children begin at school in the year when they will become five years old. Compulsory school education is from Year 1 to Year 11, which is the year in which they have their 16th birthday. They spend the Years 1 to 6 in Primary School and then move for five years from Year 7 to 11, in Secondary School (corresponding to the years K-10 in the USA). All children take exams at the end of Year 11, which is called the GCSE examination (General Certificate of Secondary Education). During Years 1 to 11, Mathematics, English and Science are compulsory subjects. Dependent on their results they can then enter Year 12 and 13 (equivalent to the American High School system) where they study ‘AS’ (Advanced Supplementary) in Year 12 and ‘A2’ (Advanced) levels in Year 13.

The main educational experience of vectors that students gain at school before studying ‘AS’ and ‘A’ levels occur during Physics and Mathematics lessons. It is therefore appropriate to look first at school text-books in the period of compulsory education to see what emphases are made in them and how these may possibly influence the teaching and learning that is taking place. Since students first meet vectors in Physics, initially at age 11, but mainly between the ages of 13 and 16, I therefore looked first at the Physics text books, and then at the Mathematics text books where they meet vectors for the first time in year 11 at age 16.

3.3 Text books analyses

This section looks at the text studied by pupils learning about vectors and aspirations of the authors who wrote the text. It shows how vectors are introduced in Physics and Mathematics.

3.3.1 How and when are vectors introduced in Physics?

In Secondary School (age 11-16) pupils meet the idea of a vector in Physics in the first year (age 11). The approach is very pragmatic and all the vector quantities lie in a line, which is either horizontal or vertical.

For example the first approach is something which, hopefully, most pupils will experience, shown in figure 3.1, taken from Heslop et al. (2000).

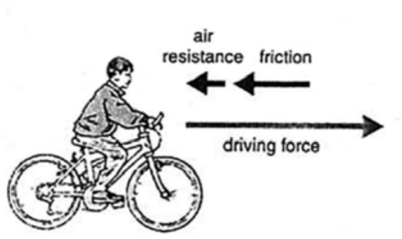


Fig. 3.1 Forces in a horizontal direction

Then pupils are introduced to the idea of the balanced forces acting in a horizontal direction, giving the resultant zero.

Vertical vector quantities are introduced, again as forces. The example, taken from the year 7 book (Heslop et al, 2000), is shown in figure 3.2.

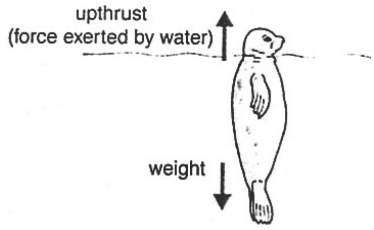


Fig. 3.2 Forces in a vertical direction

When pupils reach the age of 14, they are also introduced in Physics to other vector quantities such as displacement, velocity, acceleration, momentum and pressure. It must be emphasised that all pupils are introduced to the above ideas but differentiation occurs at age of 14 according to the students' level of ability.

In years 10 and 11 (ages 14-16), in an earlier version of the curriculum, students used to be introduced to the vector quantities involving angles, as in figure 3.3, and were then asked to draw a similar diagram for each example, which shows magnitude (size) and direction of the single resultant (overall) force. However this type of question has been removed from the GCSE syllabus (year 11), and is now introduced at AS level (year 12).

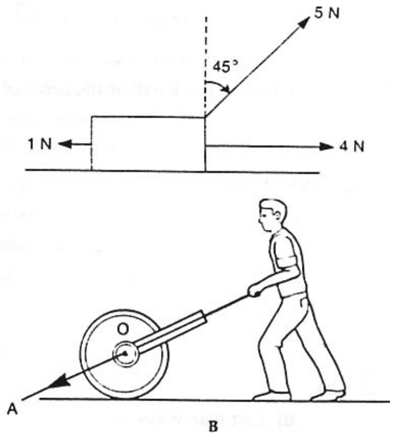


Fig. 3.3 Forces in several directions

Pupils are encouraged to make a very precise drawing of each force, measure its vertical and horizontal component, add them together and draw the resultant force based on these calculations. All these examples give pupils different physical embodiments that are intended to relate to their everyday experience.

In Physics, vectors representing especially forces are resolved as shown in figure 3.4. This uses a perspective for a vector quantity based on polar co-ordinates and suggests a vector starting from a specific point (usually a particle in Physics, or a centre of gravity of a specific object like a car a bicycle or a person), which means that a vector is ‘fixed’ to a specific point. Physics teachers do not use either column vectors or unit vectors and usually the vertical and horizontal components are written separately.

The vertical component when used with forces is usually drawn from the same position as the original force as indicated in figure 3.4.

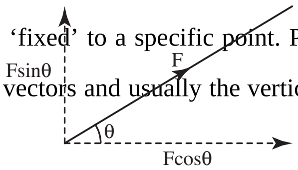


Fig. 3.4 Resolving a force into horizontal and vertical directions

This is implemented due to the belief that a force acts on the particle and therefore both components of the force should also be shown to act on that particle.

Alternatively students are encouraged to draw each vector quantity separately, measure the vertical and horizontal component and then add all the vertical components separately and all the horizontal components separately and then draw the final solution separately and measure the angle their vector quantity makes with the horizontal direction.

The teaching of vectors has been almost completely removed from the GCSE Mathematics syllabus (General Certificate of Secondary Education), which English students follow until they are 15/16. The only students who learn the idea of a vector in mathematics are Higher Course students, who are expected to obtain grades A or B in GCSE and will possibly carry on to study Mathematics at AS and A2 level in Years 12 and 13. Some of them will study Pure Mathematics with Statistics, others Pure Mathematics with Mechanics, and maybe Pure Mathematics with Discrete Mathematics. All of them will meet the idea of a vector in their Pure Mathematics in the second year of the A level study and those studying Mechanics will study them in greater detail.

3.3.2 How and when are vectors introduced in Mathematics?

When pupils are 15/16, those who are capable of achieving a higher grade in mathematics (grade A and B) are introduced to the idea of a vector in their mathematics lessons. Due to the pressure of the syllabus and time, some of them will only have one or two lessons to cover this topic. As it does not appear in the GCSE examination very often, teachers will not consider it as a priority. It may be supposed that people who design the Mathematics syllabus assume that most of the concept of vector will be assimilated by pupils from their Physics lessons.

The Mathematics Higher Course text-book, which students in this research were using in their Mathematics lessons (Pledger, 1996) gives four stages, described below, in developing the idea of vector:

1. the translation which is described as a column vector $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ as shown in figure 3.5.

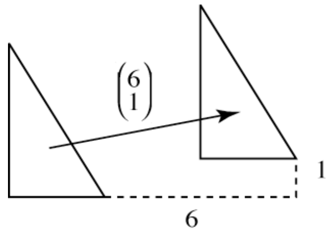


Fig. 3.5 A translation as a column vector

This is the only physical embodiment, that pupils can relate to which is presented in the Mathematical Syllabus.

2. an alternative notion is introduced to described the translation, which is $\overrightarrow{AB}$, where A is the starting point and B is the finishing point (figure 3.6)

Fig. 3.6 A translation as an arrow from one point to another

3. the third way is to describe a translation by using bold type single letters such as **a, b** (underlined when handwritten). In this case translations are simply referred to as **vectors.** The lines with arrows are called **directed segments** and show a unique **length** and **direction** for each of the vectors **a** and **b** (figure 3.7)

Fig. 3.7 Translations as arrows with magnitude and direction

4. The book then introduces the idea that the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ denotes the translation and introduces the idea of the *equivalent* vectors (figure 3.8).

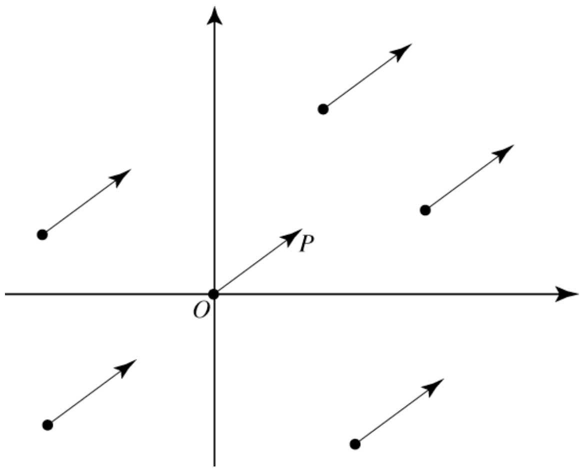


Fig. 3.8 Equivalent vectors and the special concept of position vector

The vector which translates *O* to *P*, $\overrightarrow{OP}$, is a special vector, the *position vecto*r of *P*.

In the first two stages pupils are introduced to a geometric vector with the idea that the movement and location are closely linked. In the first stage (fig. 3.5) the triangle is translated from its original position to a new position and in the second stage (fig. 3.6) , if two points *A* and *B* represent two locations, then the line segment $\overrightarrow{AB}$ represents a movement by the shortest path from *A* to *B*. The arrow shows the direction and the length of the segment represents the distance of the movement.

The fourth stage (figure 3.8) shows that the direction of the vector is represented by each or any of the parallel arrowed lines, which means that the geometrical image of a direction is not just a single line, *but an equivalence class of parallel arrowed lines*, which we call a *free vector*. However by introducing $\overrightarrow{OP}$ as a position vector or a localized vector, the book does not make it clear that if we have the situation as in figure 3.9, we would regard each of the directed line segments $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ as equivalent vectors (equal magnitude and direction). (Skemp, 1971.)

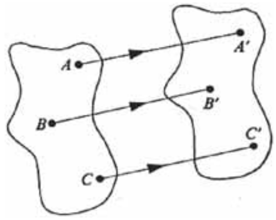


Fig. 3.9 Equivalent vectors representing the same translation

It is very important at this stage that pupils understand this concept very clearly i.e. if we are regarding all movements of the same distance and direction as being the same movement (the same *free vector*), regardless of their differences of starting point we are talking about free vector (as an equivalence class). All the other concepts pupils need to develop to be able to deal with vectors depend on their understanding of this single concept, because free vectors lend themselves to combining operations, following one free vector by another to give the concept of sum. We can always turn a free vector into a position vector, starting at origin (fig.3.8). Generally when we talk about 'vector' in mathematics we talk about a free vector.

This is emphasised in stage 3 of the development in the book.

The book introduces pupils to addition of the vectors by moving vectors parallel to their original position until they are all joined 'nose to tail' (beginning of the next vector joined to the end of the previous one, figure 3.10) as well as to the parallelogram method shown in figure 3.11.

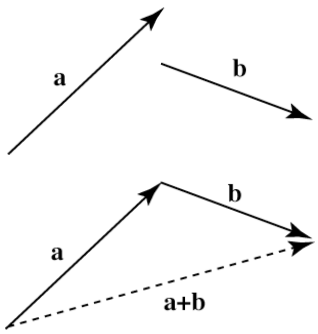


Fig. 3.10 The triangle method of addition

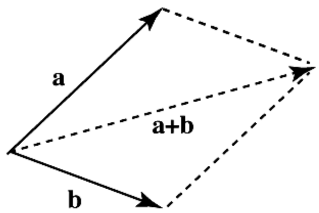


Fig. 3.11 The parallelogram method of addition

The book deals with multiplication by scalars, finding magnitudes of vectors and proving geometrical results with the use of vectors. The emphasis is on the use of free vectors and the algebraic form of *column vector* $\begin{pmatrix} a \\ b \end{pmatrix}$ to aid the calculations.

At this stage, pupils are also exposed in their Maths lessons to examples like the one below (figure 3.12) in which they have to relate vectors in the question to given position vectors.

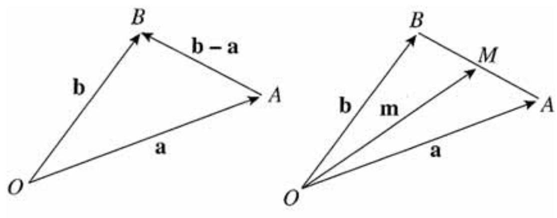


Fig. 3.12 Position vectors in geometry

They are also given examples as shown in figures 3.13 and 3.14 and asked to relate one vector as sum of others.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{AC} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$

$\mathbf{c} = \overrightarrow{OC}$

$= \overrightarrow{OA} + \overrightarrow{AC}$

$= \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a})$

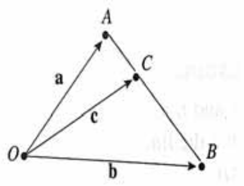


Fig. 3.13 Vector representations of geometrical positions

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. N is the midpoint of OB and M is the midpoint of AC.

Express

(a) $\overrightarrow{AB}$ in terms of **a** and **b**

(b) $\overrightarrow{ON}$ in terms of **b**

(c) $\overrightarrow{AC}$ in terms of **a** and **c**

(d) $\overrightarrow{AM}$ in terms of **a** and **c**

(e) $\overrightarrow{OM}$ in terms of **a** and **c**

(f) $\overrightarrow{NM}$ in terms of **a**, **b** and **c**.

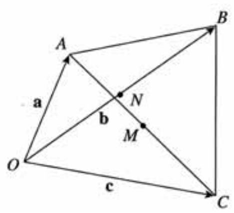


Fig. 3.14 Position vectors in geometrical figures

In the second year of A level Pure Mathematics (Year 13), which is referred to as 'A2', students are introduced to vectors in two and three dimensions. They are encouraged to change from the column vector representation to **i** and **j** representation, where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ can be represented as 3**i** + 4**j** as shown in figure 3.15.

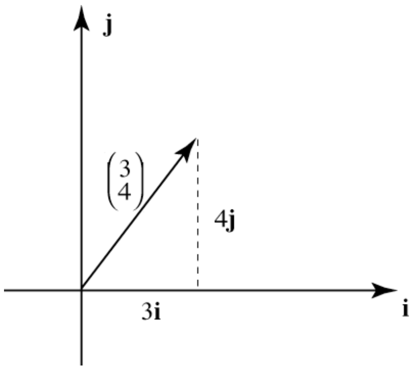


Fig. 3.15 Position vectors in terms of i and j

3.3.3 Linking the text-book sequence to process-object theory

In Physics, in the first four years of the Secondary Education the students meet vector sporadically and only in one dimension. They get to know the graphical symbol of an arrow and learn how to add vector quantities (graphically and numerically) in one dimension. After that, between the ages of 12 and 14 they are introduced to vectors in two dimensions. However, the method of operation on these vectors, for mathematical simplicity, is introduced as only in terms of the components (figures 3.4, 3.5). At this stage, they calculate the *x*-direction component and *y*-direction component for each quantity and deal with two directions separately until the final result, which they represent by drawing first the two shorter sides of the right angled triangle, and then the vector quantity as hypotenuse. This format therefore does not require the students to operate in a full two-dimensional context. The situation is simplified to what may be termed ''two times one dimension' rather than a fully-fledged two dimensional concept.

In Mathematics in year 11, as the first stage of dealing with two-dimensional vectors, teachers often follow the Physics method of solving problems with vectors in two dimensions. However in problems shown in figures 3.12, 3.13 and 3.14, students are required to have some concept of a free vector, which they tend to find, according to the interviews with their teachers, very difficult. Such problems seem to imply a huge conceptual jump.

This suggests an analysis in which the ‘two-times-one-dimensional’ stage is presented as a preliminary stage to the beginning of the two-dimensional work. Experience suggests that there are certain difficulties in moving from a preliminary stage and passing through successive stages of construction to attain the concept of free vector.

From our analysis, the mathematics text-book is written in a succession of stages that are strongly related to the *process-object-encapsulation* cycle (Dubinsky, 1991). At this first stage, the student is operating on a shape that is being translated in the plane. This shape can be considered as a ‘base object’ on which the transformation acts. This *action* can be represented by any one of a set of arrows $\overrightarrow{AB}$ of given magnitude and direction starting at some point *A* and ending at another point *B*.

At the next stage, the arrow is seen as a single entity, denoted by a single letter, say **u**. Although the move from the symbol $\overrightarrow{AB}$ to the single letter **u** seems small, it is a significant change of perspective. At this *process* stage, what matters is not the specific vector $\overrightarrow{AB}$, but just its magnitude and direction. All vectors of a given magnitude and direction represent the same *free vector*. This idea can be conceived as a mental object. Such mental objects can be added by placing them ‘nose to tail’.

At this free vector stage, the addition of two vectors **u+v** gives the same result as **v+u**. By contrast, at the specific action stage, a displacement $\overrightarrow{AB}$ moves from *A* to *B* and can be followed by another displacement *B* to *C*, so that the combined displacement $\overrightarrow{AB}+\overrightarrow{BC}$ can be achieved by moving from *A* via *B* to *C*. However, the symbol $\overrightarrow{BC}+\overrightarrow{AB}$ has no meaning as a combination of journeys in this sense, for after moving from *B* to *C*, a jump would be required to *A* to continue the second move from *A* to *B*. The move from the original idea of a translation as an action moving from one point to another is therefore quite different from the refined idea of adding free vectors. The construction of free vectors makes the mathematics subtly simpler. At this *object* stage of the APOS cycle, the student is ready to build a flexible *schema* of relationships, including such simple ideas as the commutativity of addition. At this stage the students should be able to solve not only problems like in figures 3.14 and 3.15 but also adapt their knowledge to other situations with which they may not be familiar.

3.4 Relevant examples of research into Mechanics

The research in Mechanics reveals other subtle phenomena that cause problems for students when dealing with vectors in Mechanics. This section considers the results from three specific projects that may have a bearing on the research we are about to undertake. Although the research considered moved in a different direction from our own investigations, it is important to consider the possible conceptions that can arise when students work on vector concepts. The three investigations to be considered are: ‘Students’ Conceptions about the vector characteristics of three physics concepts’ by Aguirre and Erickson (1984); ‘A Report on a Questionnaire Designed to Test Students’ Understanding of Mechanics’ by Jagger (1988); and ‘A hierarchical model of development of student understanding of force’ by Graham and Berry (1997). A brief description of each of these projects is described in sections 3.2.1 – 3.2.3.

3.4.1 Three vector concepts

This study by Aguirre and Erickson (1984) looks at the “extent to which difficulties encountered by students in the area of vectors may be attributed to their failure to comprehend some of the *implicit vector characteristics* and/or their alternative conceptions of these characteristics (alternate to that presupposed by the curriculum materials)” (p. 441). The main aim was to identify “the major constituent elements of the three vector concepts: position; displacement; and velocity; and the relationships among these elements" (p. 441). The analysis resulted in the identification of 10 *implicit vector characteristics* which are given as: reference point for stationary bodies, frame of reference, displacement or change of location, addition of displacement, subtraction of vector position, reference bodies of objects in motion, analysis of component velocities, composition of simultaneous velocities, independence of magnitudes of interacting velocities, simultaneity of component velocities. The two tasks, through which the individual students' ideas were investigated, were set in the context of familiar situations. The clinical interviews were used on a sample of 20 Grade 10 students (equivalent to English Year 11 students) to test their conceptions of vector characteristics. The results suggest that for most of the characteristics

> The largest percentage of students used *inferred rules*, which might be best called a partial description of the phenomena as viewed from a physicist's perspective. [...] when they were asked to predict the resultant magnitude of the velocity and direction of the perceived motion of the boat as it crossed the river, virtually all of the students were aware that the direction of movement of the boat would be in a direction in between those of the two contributing components. [...] But their estimates of this resultant magnitude ranged widely from values in between that of the larger velocity and the arithmetic addition of the two velocities to value in between the two velocities. [...] Other subjects tended to portray this interaction as a type of "fight" between the two components with the component having the greatest magnitude being declared the "winner.
>
> (Aguirre and Erickson, 1984, p. 452)

Another area of problem arising from the investigation of the boat question responses suggested that, "80% of the students think that the magnitude of the velocity component contributed by the boat's motor is changed in some way as a result of the interaction with the current," (p. 452).

The investigation suggested "that students possess a number of ***intuitions*** about various characteristics associated with the rather abstract and difficult topic of vectors. [...] A more detailed analysis of these inferred rules, over a variety of contexts, is required before we will be able to say much about the way in which these

intuitions can assist or inhibit instructional procedures in the area of vector quantities," (p. 453).

As the authors suggested in their conclusion, their methodology of investigating can provide a "framework for further probing of student conceptions in the area of vector quantities," (p. 453).

3.4.2 Understanding of Mechanics

This investigation was conducted by Jagger (1988) with 13 first year honours mathematics undergraduates. Many of them studied mechanics as part of their A-level mathematics and had completed a term's course on vectors in mechanics at the university. "The principal aim was to isolate the particular difficulties in understanding rate of change of velocity," (p 35), and the questions involve vector subtraction in a "pure mathematical" form. Some questions tested the students' notion of force.

In the summary, after analysing questions involving velocity and acceleration, Jagger concludes: "The problem is in moving from one-dimensional motion to motion in two or more dimensions," (p. 38). After analysis of topics related to force and motion she writes, "the pre-Newtonian view that motion implies the existence of a force in the same direction is firmly believed by quite a substantial proportion of these students," (p. 38).

3.4.3 Understanding of force

The students who study mechanics in my school use text-books written by Graham (for example: Mechanics 1, 2000), therefore research done by him is of particular interest to me. I concentrated on one of them: Graham and Berry (1997). It is a continuation of other investigations, carried out by the Centre for Teaching Mathematics at Plymouth University, into the development of students' understanding of mechanics concepts:

> The aim of the investigation was to form a set of levels, each of which would contain questions that demanded a similar level of understanding. [...] A set of criteria was selected which the questions forming the model of the development of understanding should satisfy.
> (Graham and Berry,1997, p. 840)

Some of the conclusions to their investigation were that:

> [...] their understanding of key concepts like gravity are confused, [...] students reverted to considering a constant force to be necessary to maintain the motion. [...] They also have great difficulty identifying forces and expect them to act in the direction of the motion or to be zero if the body under consideration is instantaneously at rest.
> (Graham and Berry, 1997, p.844)

Graham and Berry divided their questions into 3 levels and discovered that students passing only level one questions have sound ideas about the motion in one dimension but, for force in two dimensions, students revert to the misconception that there is a force acting in the direction of the motion. They also found that students passing their level 2 have overcome some aspects of their original misconceptions but reverted to using it in some situations. Their level three students are those who have accepted completely the Newtonian outlook on motion.

They write, that level 1, students'

> [...] difficulties arise because they are unable to identify the forces that are acting in a situation.
> (Graham and Berry, 1997, p. 847)

They suggest that:

> In order to improve students' individual understanding and promote their progression through the levels of the hierarchy they need to overcome this misconception at an early stage. It must be challenged by highlighting the weaknesses of the students' own intuitive ideas. Rectification can then take place by providing alternative explanations that the students can see overcome the weaknesses of their original ideas, explaining satisfactorily the situations used to challenge the students' intuitive ideas.
> (Graham and Berry, 1997, p. 847)

In their analysis only 23% of students have reached level 3. They suggest in their conclusion that a qualitative approach to teaching would help students to identify the forces and they would be able therefore to proceed to dynamic situations with greater confidence.

3.5 Summary of Evidence and Formulation of a Research Framework

The Physics text-books and worksheets show vector quantities always operating on a specific object. Until the end of year 11 they always act and therefore are added in one dimension. Afterwards (in years 12 and 13) when they operate in two dimensions they are resolved, for the sake of simplicity in calculations, into horizontal and vertical components and therefore operated on in what may be described as '**two times one-dimension**', rather than as single entities in two dimensions. The question arises how one can shift students' attention from working in 'two times one-dimension' to a concept of vector in two or more dimensions (Jagger, 1988).

The Mathematics text-book goes through a sequence of activities which seems to move in the direction a *process-object-encapsulation* cycle. However this cycle is not explicit, nor is it explicit in the empirical research described in the previous section which focuses instead on the difference between *displacement*, *free* and *position* vectors.

The research studies quoted also reveal how students' 'intuitions' arise from working in different contexts and how it effects their problem-solving capabilities. For example, Aguirre and Erickson (1984) talk about "*ten implicit vector characteristics"* involved in "three vector concepts: position; displacement; and velocity" and suggest that students gain "a number of ***intuitions*** about various characteristics" which need to be overcome. On the other hand Jagger (1988) says that, "The problem is in moving from one-dimensional motion to motion in two or more dimensions". Finally Graham and Berry (1997) talk about students' "need to overcome this misconception at an early stage," (p 847).

None of these researches consider the important idea of focussing on the vector concepts that are *common* to the various contexts, instead they are more concerned with the problems caused by the *differences* between them. Nor do they focus on the compression of a vector as an action into the more flexible idea of a free vector as a single mental object that can be represented by any arrow of given magnitude and direction.

As we shall see in the data of the preliminary study to be discussed in the next chapter (already published in Watson, 2002), when students meet the separate notions of displacement and force in distinct contexts, they are more likely to use the triangle law for displacement and, although encouraged in Physics lessons to use the parallelogram law for forces, they rarely do so. Indeed we find that many students use the triangle law with forces in an inappropriate way (see figure 4.16 in chapter 4) that leads to serious misconceptions. By building a coherent notion of free vector using translations, it may be hoped that the students will see the triangle law and parallelogram law not as separate rules in different contexts, but as two different ways of representing the same underlying idea. This will be investigated in greater detail in the delayed post-test analysis and the interviews in the Main Study.

From the analysis of the text-books and discussion with teachers, it was concluded that students meet the notion of vector in different contexts with subtle differences in embodiments. For instance vectors may be encountered as displacements sensed as physical journeys from one place to another, or as forces acting at particular points. In the addition of displacements, one journey followed by another is naturally interpreted using the triangle law, but the addition of forces operating at a point is more naturally represented by the parallelogram rule. In the mathematical curriculum, according the reviewed text-books, the notion of vector is first introduced as a translation in the plane and dealt with as a column matrix in mathematics, or as the separate horizontal and vertical components in physics. Both versions are linked to a picture of the vector as the hypotenuse of a right-angled triangle with components as horizontal and vertical sides. In turn this links more easily to the triangle law than to the parallelogram law.

Looking at the literature in comparison with the experience of the teachers and the text-books, it became apparent that students might be confused by trying to gain a concept of vector from many different contexts, each having different incidental properties. Aguirre and Erickson (1984) found that students fail to comprehend some of the *implicit vector characteristics* when learning from so many examples. In terms of process-object encapsulation, it does not seem that many students can move from operating on *base objects* (physical bodies, mathematical shapes) to building a *cognitive unit* from these implicit vector characteristics in the form of free vector, which in turn they could use to operate in any chosen context. A number of issues had to be determined:

- in what ways students turn the implicit properties of vectors in various contexts into misconceptions which trigger false intuitive thinking;
- what made some students able to think logically and use symbols appropriately;
- how can we may change students approach of concentrating on actions to concentrating on the effects of these actions;
- how we may help students build their vector concept into a cognitive unit which can be used easily in any context (translation, velocity, acceleration, forces, etc.);
- how can we help students use a vector as a mathematical symbol which conforms to mathematical laws of equivalence, commutativity, etc.

3.5.1 Theoretical framework perspective

According to Skemp (1971), the way to higher order thinking is through focusing on the essential properties in a given context and to filter out the “noise” (the data which is irrelevant to the required abstraction). The parts of the problem which are relevant to the solution of the problem are to be *abstracted* from the ‘outside world’ and manipulated in the ‘mathematical world’. Later, the reverse process happens “of re-embodying the result in the physical realm to give the answer to the original problem” (1971, p. 223). This cycle, according to Skemp “reduces noise” ... “and by abstracting only mathematical features it allows us to use a model which we are well practised in working," (1971, p. 223). He also says that: "The greater the *noise*, the harder it is to form the concept," (1971, p. 28).

Physics teachers try to reduce the *noise* by teaching students to work in two times one dimensional way, which is well-practised (described in chapter 4.4.4). In the case of vectors, composition of vectors by different ways of adding them and also decomposition of vectors in order to be able to apply symbolic ways of calculation is very important, just as in fractions it is very important to be able to apply the rule of equivalence to mixed numbers and improper fractions in order to be able to multiply and divide them. In vectors, to start with, the meaning is related to physical objects in the 'outside world' (translation, velocity, acceleration, forces, etc.), but then pupils are expected to develop a concept, translated into symbols, which they could operate on in a mathematical context. Eventually they should have a facility to operate, not just on those concepts, but with anything that resonates with them. The ability to work with the mathematical ideas, without the need to evoke the physical object gives the student power in solving more subtle problems. The problem is how to encourage this abstraction to occur in practice.

3.5.2 The idea of 'effect'

A major contribution to my theoretical framework occurred in a classroom discussion with a student I will call Joshua, who solved all the questions given to him in the preliminary study. During the interview Joshua explained that different actions can have the same 'effect'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation, He said "this is the same and it corresponds to the sum of the two vectors." He therefore interpreted the physical situation as mathematical, seeing the addition of two vectors having the same effect (mathematically) as the resultant vector arising from that addition.

He showed that by focusing on the *effect* rather than the specific actions involved, it was possible to get to the heart of several highly sophisticated concepts.

This concept seemed very powerful as it could be visualised as useful in other areas of mathematics. For instance, in algebra, $2(x + 4)$ and $2x + 8$ involve a different sequence of actions that have the same effect, leading to the notion of equivalent expressions.

In the case of vectors this idea could be presented graphically as in figure 3.16.

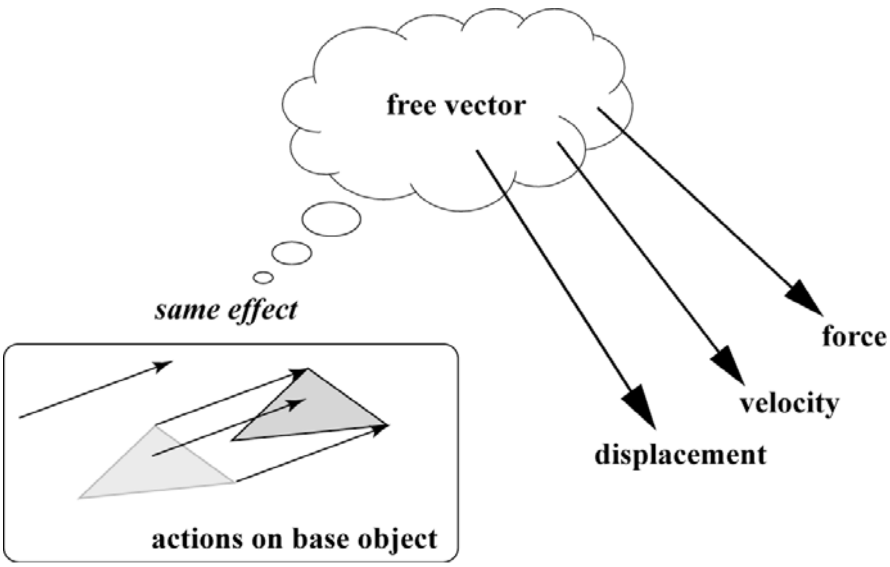


Fig. 3.16 Focusing on *effect*

The *effect* of a physical action is not an abstract concept. It can be *seen* and *felt* in an embodied sense. My idea was that, if students had such an embodied sense of the effect of a translation, then they could begin to think of representing it in terms of an arrow with given magnitude and direction. For instance, if the student's hand was moving a triangle on the table, then the arrow could be taken to show the movement of the tip of a particular finger, or thumb. The particular choice of arrow did not matter. What does matter to give the required effect is the magnitude and direction of the arrow. My idea was to use the students' physical experience as a foundation for the building of the concept of free vector and to give an underlying embodied foundation to the symbolism used for vectors building a coherent schema of meaning. For example, the addition of vectors is a simple extension of the idea that the sum of two free vectors is the free vector that has 'the same effect' as the combination of one vector followed by another. This would give embodied meaning to the technique of placing vectors 'nose to tail' to add them and would provide a foundation that showed that the triangle law and parallelogram law are just two different ways of seeing the same underlying concept, leading on to simple ideas, such as the idea that the addition of vectors is commutative.

The goal is to create conceptual knowledge with a relational understanding of the concepts rather than procedural knowledge with an instrumental understanding of separate techniques. By founding the ideas on coherent physical actions and by focusing on the notion of 'effect', the strategy is to encourage students to reflect on their knowledge and build the notion of free vector as a coherent cognitive unit in a rich schema of relationships.

This approach is also a natural extension of the foundational ideas of Piaget. It took researchers some time to realise that the important Piagetian idea of activity does not necessarily mean a physical one. As Piaget puts it: "The most authentic research activity may take place in the spheres of reflection, of the most advanced abstraction, and the verbal manipulations…," (Piaget, 1970, p. 68).

Following the literature reviewed in chapter 2, I decided to frame my work in a broad context of research including *embodied cognition* of Lakoff & Nunez (2000) — which situates the foundations of learning in real world activity, as does the embodiment of Skemp (1971) — and the encapsulation of a mathematical process into a mathematical concept through reflective abstraction, found in the work of Dubinsky (1991), Sfard (1991) and Gray & Tall (1994).

The different stages noticed in the study, which is also related to the way the text book is written, are shown below in figures 3.17 and 3.18.

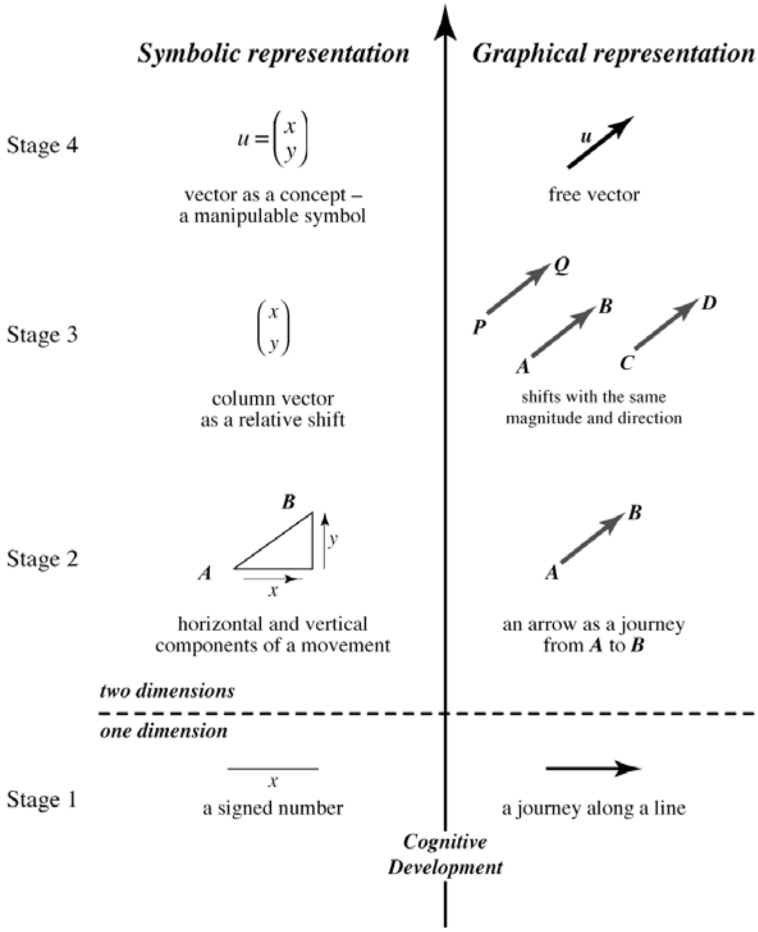


Fig. 3.17 Cognitive development of vector

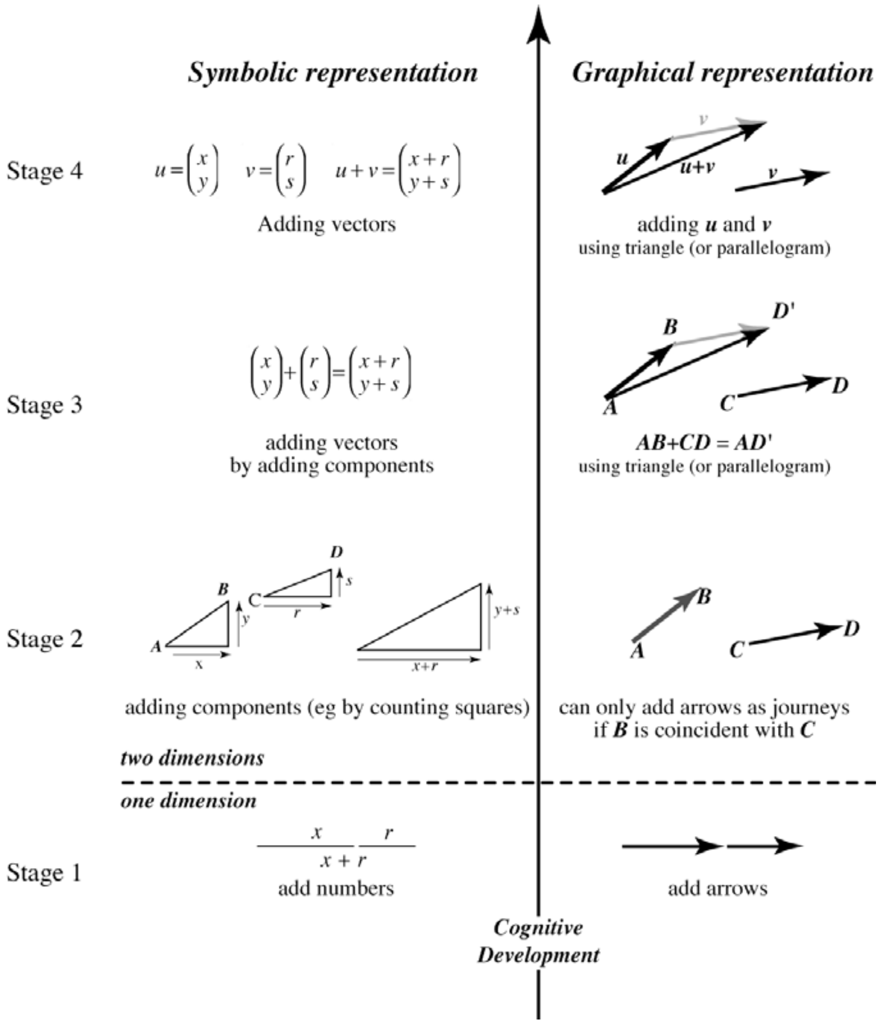


Fig. 3.18 Cognitive development of vector addition

Both of the figures show the concept-building ladder passing through stages of encapsulation. At the bottom of the ladder is the first stage, when students can deal with vectors only in one dimension and the next three levels of development in two dimensions indicate an increasing growth of encapsulation from procedure to encapsulated concept. Stage 0 is reserved for students who use their physical instinct instead of the knowledge of the vectors to answer the questions.

A further level of classification will be used that correlates the separate measures of student performance between the symbolic and graphical modes to give an overall picture of the student's development. This will use a framework that relates to the literature considered in chapter two. It has its origins in the work of Bruner, who expanded Piaget's ideas and applied them to a person's cognitive growth at any stage of life. He distinguished between three modes of mental representation: *enactive*, *iconic* and *symbolic*. He considered that these representations grow in sequence. His enactive representation begins in Piaget's sensori-motor stage and the iconic mode emerges in the pre-conceptual stage with the symbolic mode arising through language and the symbolism of mathematics. However, Bruner saw that, as each mode becomes available, all three modes are available to the individual at any age.

My interest is in teenagers who have all three modes available and, for convenience, the enactive and iconic mode of physical action and visual perception are seen to relate to physical translation and graphic representation, as opposed to the symbolic representation of vectors as column matrices and single letter symbols satisfying familiar mathematical rules, such as $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

In chapter 2, I noted how the SOLO taxonomy of Biggs and Collis (1982) incorporates both Piaget's and Bruner's idea to provide a Structure of Observed Learning Outcomes in assessing students' progress. There are five SOLO taxonomy modes of cognitive development: *sensori-motor*; *ikonic*; *concrete-symbolic*; *formal*; and *post-formal*. In particular, according to Biggs and Collis, each of these modes builds on the previous ones, so the ikonic mode incorporates the earlier sensori-motor mode, and the concrete-symbolic mode builds on these two. This fits closely with the development I am proposing in which the embodied activities refer to a combination of sensori-motor and ikonic leading to graphic representations, and the symbolic developments build on these activities.

In each mode, Biggs and Collis see the cognitive development through a sequence that they term *pre-structural, uni-structural, multi-structural, relational,* and *extended abstract*. The theoretical framework used here takes note of the analysis of Pegg and Tall (table 2.1) which suggests that the learning of any mathematical concept follows a fundamental cycle of compression related to this sequence of development in SOLO taxonomy, the APOS theory of Dubinsky, and the procedure-process-object theory of Gray and Tall (1994). In the current research study, the cycle of construction of the concept of free vector passes from pre-conceptions, via step-by-step actions (unistructural), different actions (multistructural) having the same effect (relational) to free vectors as entities (extended abstract) in both graphic and symbolic problems. These stages are as given in figures 3.17 and 3.18.

The general cycle of development underlying these stages (table 2.1) was then considered to develop a description of the stages appropriate for this study. Having simplified the SOLO taxonomy to focus essentially on embodied foundations that are represented by two modes of representation—graphic and numeric—I sought to develop an overall classification that united the developments in the two modes together.

This began with stage 0, in which students responded essentially in terms of physical intuition without any clear evidence of mathematical activity. Such a response in both graphic and numeric modes was classified as ***physical intuitive***. The next identifiable level occurs in a way that focuses on mainly symbolic or mainly graphical representations at lower stages of cognitive development. I took the decision to assign performances that attained level 1 in one of the modes but failed to reach level 2 in the other as being ***uni-modal***. This was subdivided into ***lower uni-modal*** if the activities in the higher scoring mode were at stage 1 or 2 and ***higher uni-modal*** if at stage 3 or 4. If both modes reached level 2, then the performance was categorized as ***multi-skilled***. Performances reaching at least level 3 in both modes are classified as ***versatile*** and those who attain level 4 in both modes are termed ***fully integrated***.

The following summary of this classification shows the broad correspondence with SOLO cycles (in brackets);

- **Physical Intuitive** (pre-structural) applies to students who do not abstract enough information from their physical experience to build a proper mathematical model from it and who stop using learnt procedures in unfamiliar situations.
- **Uni-modal and Higher uni-modal focused** (unistructural) applies to students can work in one mode only.
- **Multi-skilled focused** (multi-structural) applies to students who can switch between the modes dependent on the question they are asked.
- **Versatile (**relational) applies to students who use a variety of modes to answer the same question and in different physical contexts..
- **Fully integrated** (extended abstract) describes students who show a compressed concept of vector addition and show that they concentrate on an outcome rather than procedures leading to it (have the idea of the *same effect*).

The assignment of these categories is a pragmatic activity based on a careful analysis of the responses based on the theorized stages of development. Table 3.1 below shows the assignment of categories for the concept of vector in relation to the combination of symbolic development (laid out horizontally) and graphical development (vertically).

graphical mode		stage 0	stage 1	stage 2	stage 3	stage 4
free vector	stage 4	higher uni-modal	higher uni-modal	multi-skilled	versatile	fully integrated
shifts with the same magnitude and direction	stage 3	higher uni-modal	higher uni-modal	multi-skilled	versatile	versatile
an arrow as a journey from *A* to *B*	stage 2	uni-modal	uni-modal	multi-skilled	multi-skilled	multi-skilled
a journey along a line	stage 1	uni-modal	uni-modal	uni-modal	higher uni-modal	higher uni-modal
intuitive responses	stage 0	intuitive	uni-modal	uni-modal	higher uni-modal	higher uni-modal
		stage 0	stage 1	stage 2	stage 3	stage 4
symbolic mode		no response	*x* a signed number	horizontal and vertical components of a movement	$\begin{pmatrix} x \\ y \end{pmatrix}$ column vector as a relative shift	$u = \begin{pmatrix} x \\ y \end{pmatrix}$ vector as a concept D a manipulable symbol

Table 3.1 Development of the vector concept, combining graphic and symbolic

Table 3.2 shows the corresponding assignment of categories for the concept of vector addition. This again relates to the combination of symbolic development (laid out horizontally) and graphical development (vertically).

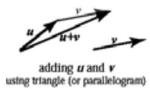
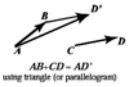
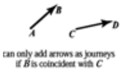
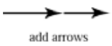
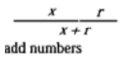
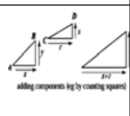
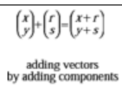
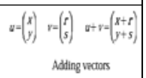
graphical mode	 adding u and v using triangle (or parallelogram)	stage 4	higher uni-modal	higher uni-modal	multi-skilled	versatile	fully integrated
	 AB+CD=AD′ using triangle (or parallelogram)	stage 3	higher uni-modal	higher uni-modal	multi-skilled	versatile	versatile
	 can only add arrows as journeys if B is coincident with C	stage 2	uni-modal	uni-modal	multi-skilled	multi-skilled	multi-skilled
	 add arrows	stage 1	uni-modal	uni-modal	uni-modal	higher uni-modal	higher uni-modal
	intuitive response	stage 0	intuitive	uni-modal	uni-modal	higher uni-modal	higher uni-modal
			stage 0	stage 1	stage 2	stage 3	stage 4
			intuitive response	 add numbers	 adding components (eg by counting squares)	 adding vectors by adding components	 Adding vectors
			symbolic mode				

Table 3.2 Development of vector addition, combining graphic and symbolic

The lowest categories in each table show students operating on basic objects. The highest categories of each of the tables show that students are able to compress their knowledge to operate with vectors as cognitive units in any situation.

Students might be able to operate in one mode (symbolic or graphical) only and achieve a high stage at that level (higher uni-modal) or they can operate in both modes using them to reinforce their answer to a particular question or use a flexible choice of different modes in different questions (versatile). Students who can use both modes comfortably to any type of situation will recognise the commutative law of addition of free vectors. This is the highest stage of cognitive development for the purpose of this research, which does not include developments into formal mathematics based on axiomatic definitions and formal proof.