Chapter 4

Preliminary Investigations

4.1 Introduction

In this chapter I discuss preliminary investigations into students' difficulties in my classroom and in consultation with other teachers and interviews with students. Topics of importance include:

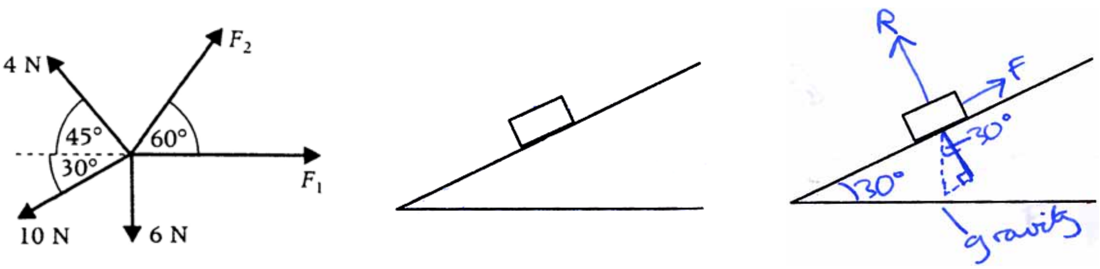
- my investigation of the way vectors are presented in Physics and Mathematics classes at the secondary level;
- my observations, as a teacher, of the problems students have in dealing with vectors in Mechanics (from mathematical and physical points of view) and Pure Mathematics, together with discussions with other teachers to check if they have different experiences from mine;
- the three researches described in section 3.2 (Aguirre and Erickson, 1984; Jagger, 1988; and Graham and Berry, 1997);
- the theoretical framework gained from cognitive science literature of embodiment, and the mathematics education literature focused on the use of symbols representing both process and concept;
- the development of a method of assessment of cognitive development stages to formulate a framework to interpret students' responses.

As it was not obvious at which stage a problem was occurring, and because the claims from other researches needed to be tested, the preliminary investigation began by investigating a question from the Mechanics text-book which, from experience, students found difficult to solve. Students were given the question and then, after analysing the range of responses, some students were interviewed to investigate how they went about solving the problem. Due to the claims of the researches described in chapter 3, students were also questioned on Newton's three laws, to check if this had any bearing on their responses. This investigation is described in detail in Watson (2002). Some of the results presented in that paper are shown below.

4.2 Preliminary empirical investigation

The research has been conducted in a Comprehensive School with good average results with the Sixth Form Centre fifth in the national tables comparing schools in terms of the Value Added (the increase of the level which students achieve in the centrally controlled National Curriculum).

Given a problem solvable by using horizontal and vertical components such as figure 4.1 (a), 25 out of 26 students were able to solve it. However, given a more complex physical problem such as that in figure 4.1 (b), asking the student to mark the forces involved with an object on a rough sloping plane, only 4 out of 26 students were successful. In interviews, it transpired that several students, who used the triangle law to draw a picture as in figure 4.1 (c), used the triangle of forces to mark the components; because the force parallel to the plane is drawn well below the object, it did not seem to be acting *on* it and was ignored.



(a): find F_1, F_2 (b): describe & mark forces (c): forces as marked

Fig. 4.1 Two questions on forces (a slope)

`Five students—who gave varied responses, from one not answering the question at all to the one giving a correct answer—were interviewed. These interviews indicated that even students who did not answer the question knew that the object will slide if there is a resultant force, acting on it. The problem was that, according to their analysis of their own drawings, the resultant force was acting in the in a wrong way — up the slope.

Most students, as in drawing 4.1c, resolved the weight in parallel and perpendicular directions (two components of a vector), drawing the parallel component as part of triangle with the relevant component well away from the surface. The two components were calculated correctly by the majority of students, showing that they are trigonometrically competent. It seemed that they could not proceed any further because although, they knew from their correct *intuition* that the body will either stay where it is (if the forces are in equilibrium) or will slide down (if there is a resultant force), they could not find a force which performs the expected action.

The procedure of drawing the components of weight were well-learnt but not understood. The lack of arrows (correct use of symbols) on the weight and its components could be to blame but a more likely source of difficulty was the fact that the parallel component did not seem to operate directly on the object. The students were able to explain in the interview that the perpendicular component balanced the reaction force *R* and therefore “the object will not sink into the surface or fly off it”. However, it seemed that the only evident force parallel to the plane was the frictional force *F*.

To investigate further the reasons underlying the original problem in figure 4.1 (c), a question was given to students showing a body on an inclined plane, as in figure 4.2(a). Figures 4.2(b) and 4.2(c) were said to represent the ways in which two students James and Chris split the weight W into components W_1 and W_2.

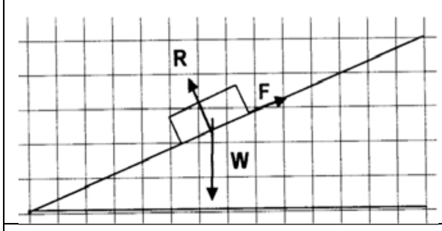
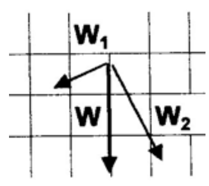
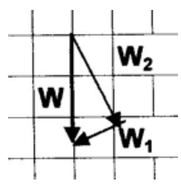
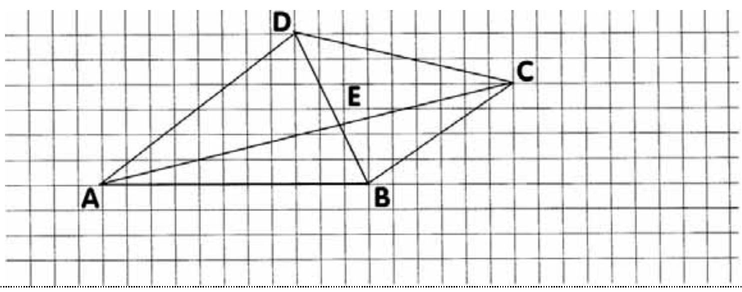
		
(a): a body on an inclined plane	(b): James’s picture	(c): Chris’s picture

Fig. 4.2 Preliminary study questions

The students were asked: “Are either or both of James and Chris right?” The question was given in this specific way to take the pressure off the students so that, rather than giving their own answer, they were asked to comment on somebody else’s responses.

The 23 students beginning the course in year 11 gave a variety of responses, 11 said both were right, 4 chose fig 4.2 (b), 1 chose fig 4.2 (c) and 6 said neither. These results indicated that although half of the students seemed familiar with the equivalence of the triangle and parallelogram laws by saying that (b) and (c) were both right, the other half responded differently. Since 25 out of 26 students could solve a problem presented in figure 4.1 (a), they seemed to be familiar with vertical and horizontal components. It might have been possible that the context in which the question was asked caused the problem, which may have occurred with the 5 students who had chosen only one of (b) or (c) in figure 4.2.

To test the student's ability to deal with vectors graphically, without any physical context being involved, they were given the question shown in figure 4.3 which was of a type they encountered in Year 10. Part (i) is a natural triangle problem with the vector *AB* followed by *BC*. Part (ii) could be solved either with the parallelogram or the triangle law, however students had to draw the additional lines, which they had not been expected to do in their text book exercises. Part (iii) is more subtle. If they were aware of the commutative law of addition of vectors they could add them as $\overrightarrow{CA}+\overrightarrow{AB}$, however if they saw the addition as 'journeys' this would not make sense to them. On the other hand they could have treated the vector as free and move them 'nose to tail'. A third option was to answer numerically.



Show clearly a vector equivalent to:

(*i*) $\overrightarrow{AB}+\overrightarrow{BC}$

(*ii*) $\overrightarrow{AB}+\overrightarrow{AD}$

(*iii*) $\overrightarrow{AB}+\overrightarrow{CA}$

Fig. 4.3 Testing the visual sum of two vectors

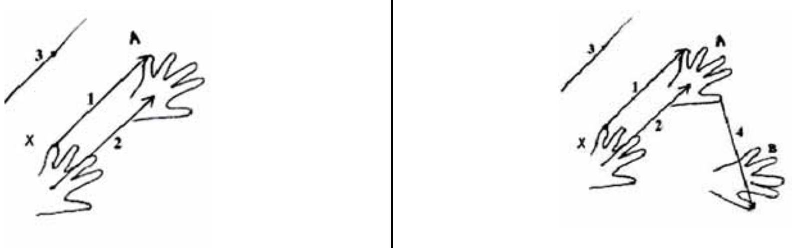
In the test, *all* the students were easily able to cope with the first sum $\overrightarrow{AB}+\overrightarrow{BC}$. However, parts (ii) and (iii) were more problematic and only 3 students out of 23 managed to answer at least one of these questions; all of these who responded correctly solved the problem numerically. This suggested that the students did not grasp the idea of a free vector as a cognitive unit that can be operated on in any context; they were only able to cope with either a simple mathematical problem, as in figure 4.3 (i) or simple physical problem, as in figure 4.1(a).

The research literature discussed in chapter 2 suggests that students should be able to construct a meaning from the experience from the physical world (Piaget, 1985; Bruner, 1966; Lakoff and Johnson, 1999; Lakoff & Nunez, 2000). However concern is also expressed about the "prototype effect," (Rosch, in Lakoff, 1987) and "interpreting words and gestures differently," (Jaworski, 1994).

With all these factors in mind I decided, as suggested by Jaworski (1994) to perform "the activities in which learners participated and encourage them to be mathematical, that is to act as mathematicians by mathematising particular situations created by their teacher" and by including group work and reflective plenaries to encourage learners to "share perceptions with each other and with the teacher", and therefore to make sure that "their ideas became modified or reinforced as common meaning developed."

Two groups were chosen in Year 12, specified as experimental and control, where the experimental group was taught using physical activities and reflective plenaries, which the control group were taught by following the text-book. The students in both groups were tested again and assessed according to the same method as before. A selection of students from each group was also interviewed.

In dealing with the specific vector problems, the students in the experimental group were encouraged to participate actively by shifting a hand placed on the paper and draw the vectors which could represent the translation as shown in figure 4.4(a); then a second translation represented from a different finger as shown in figure 4.4(b). Then the students were encouraged in plenaries to discuss different vector representations of the translation and the way the resultant movement can be represented using vectors in their drawings.



(a) (b)

Fig. 4.4 Embodied action

After two weeks the students were again given the questions presented in figures 4.2 and 4.3. When we considered those students who were able to solve all three problems, we obtained the data in tables 4.1

Year 12	Embodied (N=7)	Standard (N=16)
All 3 correct	5	1
Other	2	15

Table 4.1 Effect of embodied approach in reflective plenaries

Those following an embodied approach had more success answering the questions.

Interviews with six selected students, three from each group, confirmed that students following a standard course had problems adding two vectors that did not follow on one after the other, especially in cases where they were joined head to head. In the latter case, two out of three students thought that two vectors pointing to the same point would have resultant zero, because they would cancel out.

4.3 Summary to preliminary empirical investigations

The study so far has revealed the complexity of the meaning of vectors as forces and as displacements and the subtle meanings that are inferred in differing contexts. Studies in science education have attempted to build a classification of misconceptions without clearly identifying the underlying problems. Our approach is to develop a pragmatic method that will work in the classroom. One aspect is the use of conceptual plenaries, which are already becoming part of the formally defined curriculum in England. The other is to continue to develop a theory that links physical embodiments to mathematical concepts via a strategy that focuses on the effects of actions. Our experience shows that such an approach can be beneficial in the short-term and we are continuing our practical and theoretical developments over the longer term.

As a next stage of preliminary investigation I gave selected students modified questions, based on the above research. However after analysing the results and interviews, I decided to look more at students understanding of an idea of a vector in its different contexts, rather than only the Newtonian problems students are faced with in mechanics.

Some of the questions in that stage of investigation were set on the squared paper as in figure 4.3. After looking at the results of the test and the interviews this idea was dismissed as students simply counted squares to add the components of the vectors and did not show any conceptual thinking.

The preliminary study also seemed to show some evidence for the work of Dubinsky (1991) and Sfard (1991) of *process-object encapsulation* and the theory of Gray & Tall (1994) that students use such symbols both as processes to *do* mathematics and as concepts to *think about*. However there was evidence that many students do not seem be able to use the concept of equivalent vector or free vector in every context. It is as if, for some students, the complication that occurs in a specific context triggers 'false intuitive' reasoning and removes the ability of logical/mathematical thinking. However, when a given problem is presented in an easier way or they are reminded during the interview about the theory (for example of addition) their power using procedures is often correctly recalled. The problem seems to be complicated by the fact that the students are more concerned with remembering to carry out a given procedure rather than reflecting on its total effect. In terms of Dubinsky's theory, the students seem to be focusing more on the *action* stage (of externally taught sequences of steps) than the *process* stage (where the process is interiorised as a whole).

In the teaching experiment I decided to focus students' attention on the underlying mathematical concepts that I believe to be theoretically simpler even though students often find them challenging. My approach was based on the embodiment of the ideas initially as physical actions and then to focus on thinking of the actions as processes that are symbolised and considered as thinkable objects as expressed in APOS theory. However, although this theory starts with *actions*, the starting actions must act on already known objects. In the case of vector as a transformation in the plane, the action operates on figures in the plane that are translated. My research therefore begins with the 'base objects' that the initial actions act upon, with the initial learning strategy based on how the actions transform the objects. In the case of vectors as translations, a base object might be a triangle on a flat table and the actions may be the translations that shift the triangle from one position to another. The essential problem, which has proved problematic in many settings in the literature (eg Cottrill et al, 1996, p.187), is how to achieve the full development from the initial focus on the actions to the final encapsulation of the ideas as mental objects.

4.4 Relating empirical evidence to theoretical framework

By comparing the students' responses to the questions posed in the preliminary investigation it may be concluded that students reach different stages presented in the mathematics text-book and, dependent on the stage achieved, they can solve questions of varied difficulty. They also often seem to have a preferable mode of operation (graphical or symbolic). They might be at a different stage of development in understanding the vector concept than in understanding the idea of vector addition.

The examples of the way these levels should be understood in terms of students' responses and the way they were awarded will be considered in detail in the data analysis in chapter 7. Many researchers indicate that is it easier to show what students cannot do rather than what they think and imagine (see, for example, Sfard, 1991). To complement the quantitative data obtained from the questionnaires, the assignment of stages will be triangulated with qualitative methods arising from interviews with staff and students.

The preliminary study shows that there is a difference to students' development when they are exposed to the experimental lessons in which the emphasis was directed to compressing the embodied actions into process by focusing on the notion of *effect* (if two actions have the same effect then they are considered as giving the same process). It was therefore decided that for the Pilot and the Main studies one group of students, which we will call the experimental group will be involved in lessons in which they will move a hand across the paper as well as push objects across the paper with a hand and focus on the effects of these actions. In the follow-up plenaries, the students will discuss the idea that two physical actions of movement, from point *A* to *B* and then from *B* to *C* (one following the other), are mathematically equivalent to the physical action of single movement from *A* to *C*.

The students will be encouraged in plenaries following the embodied exercises to reflect:

- that the physical action in the embodied world can be modelled mathematically as a symbolic procedure, and on the effect of that procedure;
- that the same mathematical meaning underlies different physical contexts (particularly journeys and forces);
- and appreciate that the mathematical process conducted through different modes of operation (symbolic/graphical) gives the same effect even though the representations may be different.

The researcher hypothesises that the notion of 'effect' is an important stepping-stone in a cognitive development that links the concepts in the embodied, symbolic, (and later the formal) worlds of mathematics. It was conjectured that this will correspond to the cognitive compression of mental processes into thinkable objects in which processes become concepts, and in which the symbols will allow the students to use their knowledge equally successfully in different contexts. (For instance, the notion should later lead to the notion of equivalence relation in the formal world.)

The teaching experiment will be aimed at students giving meaning to concepts in the embodied world, and then sharing their experiences with their teacher (as mentor) who will guide them to express their ideas to each other in ways that enable the embodied concept to be converted in meaningful and flexible ways into symbolic and formal ideas.

In the experimental stage, the rigorous pattern of the Numeracy Strategy will been used which specifies that each lesson should have three stages: starter, core, and plenary. During the starter activity, the teacher sets the scene with the whole class for the main part of the lesson. During the core part, the students work in groups or on individual tasks, and the final plenary reflects on the ideas met in the lesson and makes connections between them. In years 12 and 13 this pattern is usually followed only to a limited extent.

The hypothesis of the researcher that this approach should help students move to a higher levels of cognitive development and retain the conceptual awareness, will be tested through the three tests, staged at intervals: one before the experimental lessons, another soon after and the third after half a year. The experimental group's responses will be compared with responses of students in another group which will not participate in the experimental lessons and which we will call the control group. The interviews conducted after the first and second test are intended to clear any uncertainties about students' test responses and show if students in the experimental group will use more mathematically based language compared with the control group students, independent of the context they will work in.