Chapter 10
Summary and plans for the future research

10.1 Introduction

This research was designed to test if the cycle of process-object encapsulation to form the concept of free vector can be enhanced by concentrating on *effect*. The intention was to build on students' intuitions to get 'real' understanding of the vector concept and to encapsulate it as a symbol of a free vector which can be operated on as a cognitive unit.

The sample of literature from science and mathematics education showed how complicated physical intuitions can be (Aguirre & Erickson, 1984, Jagger, 1988, Graham and Berry, 1997, Dubinsky, 1991). Aguirre and Erickson found various *vector characteristics* used in different contexts and discovered that most students used partial descriptions, mainly based on intuitions related to these characteristics when describing and dealing with different physical phenomena. Their research concentrated on students' conceptions in different areas of vector quantities and suggested further investigation on the same basis. Jagger's (1988) research also concentrated on studying difficulties students had with vectors in different physical contexts. She found that the change from one dimension to two dimensions proved to be a significant problem as well as lack of understanding of the Newton's laws of motion. Graham & Berry (1997) similarly concluded that students seem to have problems with different physical concepts and Newtonian laws, which acted as an obstacle to their use of mathematics. They suggested as a remedy an approach that challenges students' 'intuitive ideas'. On the other hand, the research of Dubinsky (1991) from a mathematical viewpoint showed that the cycle of process-object encapsulation is difficult to complete, with students often reaching only the process level and failing to conceptualise the process as a mental object.

Skemp (1976) suggests that the mathematical idea should be built, not by working with several different contexts at once, but by focusing on one particular context to develop the mathematical concepts in a way that can then be applied to other contexts. In the case of constructing the mathematical concept of vector, the science education literature shows ample evidence of a range of 'false intuitions' that may arise. In choosing a specific context to work in, I chose to start, not with forces or journeys, but with the idea of physical transformations used in the mathematics text-book.

The main goal for my research is to seek a solution that enables students to reach a level where a free vector is encapsulated as a flexible mental object. The proposed solution, tested in this research, is to begin in the single context of a vector as a transformation, to focus on the *effect* of the transformation, to provide students with a focus for the construction of the concept of free vector.

To encourage students to construct meaning for themselves, in a way that is consistent with the mathematical theory of vectors, the lessons began with physical activities in which students performed the action of translating a triangle on a table. The triangle functioned as a 'base object' on which the translations acted and, by focusing on the effect of the translation, students could gain experience that any arrow of a given magnitude and direction could be used to represent a translation of that magnitude and direction. The concept of an arrow as a free vector was then made the focus of attention and the addition of 'free vectors' by moving them 'nose to tail', giving a result that has the ***same effect*** as the action of following one vector by the other. The activities looked at different ways in which the vectors could be added (for example using the triangle method or the parallelogram method) to see their equivalence.

The students' own construction of the notion of free vector was supported by activities and discussions in reflective plenary sessions. The idea of the reflective plenaries has arisen from work of Barbara Jaworski (1993) who implied that after activities in which students participate, the teacher should create the situation in which (s)he can enable them to construct meaningful concepts. This proved also to be advantageous as it linked with the idea of using plenaries in the English National Curriculum. In these plenaries, students were encouraged to build a meaningful concept of free vector as encapsulated object that they could operate on in different contexts, mathematical as well as physical.

The Preliminary Investigations helped to build the main hypothesis formulated for testing in the main study:

> **Main Hypothesis**: Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

This was developed from my instinctive feeling that if students were able to concentrate on the simplicity of mathematical ideas instead of the many complications connected to different contexts, they then would be in a better position to solve problems occurring in those contexts.

The goal of the research was to find a strategy that would enable students to concentrate on the simplicity of the mathematical idea of vector instead of considering difficulties and variations in different contexts using vector quantities.

After a review of relevant research (chapter 2), the research framework to be used at the outset was outlined in chapter 3.

The empirical research consisted of three stages: an initial exploration of ideas that seemed relevant in a preliminary classroom study (chapter 4); the methodology and methods to be used (chapter 5); a pilot study to test out the teaching experiment and the design and analysis of the questionnaire to produce refined hypotheses and methodology for the main study (chapter 6); the hypotheses were tested through the analysis of the results of the questionnaire in a pre-test, post-test and delayed post-test (chapter 7), to be triangulated with interviews with teachers (chapter 8); and interviews with the students (chapter 9).

10.2 Theoretical framework

The strategy evolved from the Preliminary Investigations was to work in an environment which enables students to have the potential of focusing on the essential properties. To be able to work in such situations and to move from activities to essential mathematical concepts, a fundamental focus on specific ideas has to occur, which should lead to the essential compression of knowledge. This was encouraged in two ways:

- o by embodying actions and focusing on the *effect* of these actions;
- o by assigning a symbol to the effect to enable it to be conceptualised as a single idea — a cognitive unit.

It was hoped that the power of this essential idea can be related to other contexts where the focus is now on the essential properties rather then the incidental details that previously caused difficulties.

In the case of vector this was done through the translation of an object on a flat table; the students were encouraged to not concentrate on the movement of the object or some particular point on that object but on the *effect* of the movement.

The movement of the particular point on the object from A to B can be represented by a particular arrow to which we can assigned a symbol $\overrightarrow{AB}$, however the essential idea is the effect of the movement which can be represented by *any equivalent arrow* having the same magnitude and direction. In this way it becomes possible to imagine that these equivalent vectors operate as a single entity that represents the more subtle concept of free vector. A bonus is that the combined effect of one free vector followed by another can be represented by placing arrows representing the vectors 'nose to tail' to give the sum as the single free vector that has 'the same effect'. If we can free ourselves from the physical contexts, such vectors can be joined together in any order to give a unique result. In particular, the triangle law and the parallelogram law are two different ways of seeing *the same idea* and can be used interchangeably. The theoretical framework was design to test the hypothesis claiming that if students participated in the experimental lessons and meaningful discussions, they should be able to use vector flexibly, as mathematical symbol, and retain their knowledge for a longer period of time.

The hypotheses, discussed in detail in section 10.2, were tested through three tests. The comment on the results of the tests and interviews will be discussed in section 10.3.

10.3 Themes of the testing

Three parts of the main hypothesis, described already in chapter 7, were developed and tested. These were:

> **Hypothesis 1**: Students, who were involved in experimental lessons, are expected to rise through the cognitive stages further than students who are not exposed to the experimental lessons.
>
> **Hypothesis 2**: Students who were helped in building a concept of a free vector are expected to be more able to:
> **(a)** add vectors in singular cases, not just generic ones;
> **(b)** use free vectors independent of the context;
> **(c)** realise that the commutative law applies to vector addition.
>
> **Hypothesis 3**: Students who can concentrate on the *effect* of actions rather than actions themselves are more likely to build the concept of free vector as a cognitive unit, which can be used by students after a longer period of time and not only just after the experiment.

The interviews, as described in chapter 9, were intended to gain a greater insight into:

- students' use and flexibility of language when discussing problems connected with vector addition;

- students' focus of attention at any given time (whether it is on actions, or procedures or on the effects of those actions and procedures);
- the way in which different contexts affect their thinking;
- their flexibility in dealing with different modes of operation (graphical/symbolic).

10.4 Testing Hypothesis

The three hypotheses were tested three times: in the Preliminary Investigations, which helped to build the methodology; the pilot study which tested the methodology; and the main study, which proved that there were significant positive changes in the experimental group, compared to no significant changes in the control group. All three studies indicated positive change in students who have undergone the experimental lessons. During the main study, two groups of students were tested three times throughout year 12. The first test (pre-test) was conducted at the beginning of the year. The second test (post-test) was conducted one month after the part of the Mechanics course involving addition of forces finished and two months after the pre-test. The third test (delayed post-test) was conducted a year after the pre-test, when students came back from their summer holidays. They were analysed using methods developed in chapter 4 and detailed in chapters 5, 6 and 7.

The significance of the changes in the stages of the cognitive development that were achieved by students, was determined using the two-tail t-test. The other comparison was done using the scatter graphs and the chi-squared test. This test looked at the comparison of the proportions of students in two different areas of the graph: lower lever area included *intuitive* and *uni-modal* categories; and the higher level area included *higher uni-modal*, *multi-skilled*, *versatile* and *fully integrated* categories. The t-test taken for the graphical changes between the post-test T1 and the delayed post-test T3 show highly significant changes for Group A ($t=3.83$ at $p<0.01$) and no significant changes for Group B. The changes for the symbolic mode were not significant for either group. When we triangulate the overall responses to all three tests and teachers' comments together with the students' interview responses, we can see that only some students at the beginning of the year even considered answering the questions graphically. The students seemed to realise that the test (without grids) was answered more efficiently graphically and attempted mainly to do so even if their graphical competence was not adequate to do so. The symbolic answers were given mainly as alternative responses but rarely as main responses. This is in all probability the reason why in all the tests the changes in the symbolic responses for both groups were not significant.

These results provide evidence for hypotheses 1 and 3. The students who were involved in the experimental lessons rose through the cognitive stages further than students who were not exposed to the experimental lessons and their conceptual understanding worked after a longer period of time and not just immediately after the experiment.

Differences occurred in the case of singular (hypothesis 2 (a)) questions where, for example, students from Group A were able to cope better with two vectors meeting at one point. The t-tests performed on students' changes in the stages of the graphical cognitive development between the pre-test and the delayed post-test show that Group A underwent highly significant positive changes ($t=3.13$ at $p<0.01$) while the changes in Group B were not significant. These results support further hypothesis 3 that Group A students' conceptual knowledge of vector addition was more firm by the time of the delayed post-test and they could apply it more flexibly, even in the singular cases. The chi-squared test showed there was no significant difference between the two Groups in the delayed post-test. However if we consider that there was a significant difference with $\chi^2 = 4.24$ ($p<0.05$) in the pre-test in favour of Group B and in the delayed post test χ^2 changed to 2.32 in favour of Group A, we can see that the positive change has occurred in favour of Group A. In fact Group B has not changed and only Group A has.

The highly significant positive changes also occurred in case of Group A when responding to the questions set in two different contexts ($t = 8.71$ at $p<0.01$) The Group B also improved but less significantly ($t = 2.17$ at $p<0.05$). The chi-squared test also shows a significant difference which favoured Group B in the pre-test ($\chi^2 = 5.24$ at $p<0.05$) to the significant difference which this time favoured Group A in the delayed post-test ($\chi^2 = 4.84$ at $p<0.05$).

This shows that, on the whole, Group A made much more significant improvement than Group B in their stages of cognitive development as far as the singular questions and the different contexts questions are concerned. From the students' post-test and the delayed post-test responses it also became evident that students in Group A treated these questions in a more 'mathematical' way. The substantial number of them used their knowledge of free vectors in addition with confidence.

These results support hypothesis 2 (a), that the experimental Group A, in comparison with control Group B, gained conceptually from the experimental lessons in the context of vector as force and sustained their knowledge between the post-test and the delayed post-test. The difference between the groups changed from Group B being significantly higher in the pre-test to Group A being significantly higher in the delayed post-test. It is relevant that there was no significant difference between the groups in the post-test, which emphasises the long-term effect of the experimental treatment.

In addition to the results from the quantitative analysis the interviews also showed that there is an apparent difference in the language which the students operating at different levels of cognitive development use to describe their responses to the test questions (chapter 9). The analyses indicate that students operating at lower cognitive levels use procedures without using the concept of the free vector, while the students operating at higher cognitive levels developed the concept of vector as a cognitive unit. This gives the additional qualitative support to hypothesis 3.

For example one of the high attaining students from Group A, when asked how he tackled the singular case, in which two vectors met at one point, he said: “I was sliding the vectors so one is at the end of the other, so that they are nose to tail, and then drew a resultant. [...] I worked out the length and direction, did them in i and j direction and added them together.” When asked how he answered two questions set in different contexts (forces and displacement), he answered: “They are the same. You could do them in the **i** and **j** direction and add the together, or you could draw them so they are nose to tail and draw the resultant.” When asked how he approached the singular question at the end of the test, which many students found difficult even to start, he responded: “I didn’t know how to do this and this was like a second thought [...] I was making my own way of doing it.” His answers show that he has built a cognitive unit, which he was confident to apply to an unfamiliar situation. He also implied that every time he looked at the *result* of the addition, which meant that he concentrated on the *effect* of the addition, in both symbolic and graphical mode.

On the other hand another student, this time from Group B, who just put two vectors together but did not draw the resultant, when asked how he went about answering the question responded: “I did not know what you meant by ‘add the two vectors’, so I assumed it was put them together as arrows.” He obviously concentrated on the procedure of addition and not on the *effect* of it. When asked to explain a bit more what he understood by addition, he answered: “I understand the addition as showing the total movement.” When asked how he tackled two questions, one asking him to draw and add three forces and another to draw and add two displacements, he answered: “Apart from the fact that there is an extra force in the first one, they are exactly the same.” He concentrated in both situations as if they applied to forces and answered them in that context, but not as free vectors in a mathematical context, in the manner that the previously described student did. When asked if he noticed the contexts are different he said: “Ah […] the forces are not necessarily vectors I don’t think they are movements.” He related to vectors as movements but since he knew that forces acting on an object do not have to cause a movement, he therefore did not think that forces are vectors. He did not build a notion of vector into a cognitive unit. To him it was a different symbol when used in the different contexts of forces and journeys. There was no indication in the test that he knew that the addition of vectors is commutative.

10.5 Summary of testing theory

The quantitative analysis in Chapter 7, firmly confirm the main hypothesis by providing statistical evidence to support hypothesis 1, 2 and 3. The first part of the qualitative analyses (chapter 8) shows that the teachers clearly understand the kind of mistakes that the students might make, particularly the two mathematics teachers. The physics teacher differs, probably because she was looking at how students would respond in the Physics context, while the Mathematics teachers were considering how students would respond in Mechanics context. Students might try to adapt their responses to the subject they have to operate in, as they might try to respond in they way they think the teacher wants them to respond.

Certain things came clear which the original theory did not consider explicitly, for example when we look at the way some students add two vectors together ($\overrightarrow{AB}$ and $\overrightarrow{BC}$), they move the beginning of the vector $\overrightarrow{BC}$ to the end of the vector $\overrightarrow{AB}$ and leave it. They treat this addition as showing the journey from A to C via B and therefore as far as they are concerned the task is completed. They see it as showing a journey and not as vector addition using equivalent vectors. To triangulate this with the teachers' comments: one teacher says that "they forget to put the arrow in," however it is possible that they do not forget but they just are not at the level where they understand the purpose of putting the arrow in. If they are told to remember it may not help them to understand the concept but might help them to get the good marks in the exam.

The theory developed in this thesis, on the other hand, suggests the alternative to warning students what to do or not to do. It says that if we involve them in specific physical actions on vectors and mentor them in reflecting on the effect of these actions (correspond to the idea of free vector), then the students will have some personal experience, from which they will be able to sense what it means to move the free vectors around. If they afterwards see the vectors on the paper they are more likely to be able to imagine moving them around. The other students might just have the experience of being told to place them one after another and place another arrow from the beginning of the first one to the end of the second one, without developing a concept of a free vector. These two approaches to teaching might have the effect of how long students remember how to solve vector problems. From the test results it shows that the embodied approach followed by reflecting on actions helps the teaching to have more permanent effect on what students remember. It may very well be that the non-verbal action of physical movement and the sensory and visual effect of this movement is more deeply entrenched in their personal psyche. Thus, over the long term, it fits more naturally with their thinking processes and is enhanced as time goes by. It may also be that, being non-verbal, the students find that they can 'do' the operation naturally and successfully, and yet, when interviewed, they may not be fully able to verbalise what they are doing.

Chapter 8 shows also another discrepancy with the theoretical framework developed in the Preliminary Investigations, which involves use of the parallelogram law of addition. The theoretical framework suggested initially that forces were added using the parallelogram rule of addition and displacements were added using the triangular rule, although it might be true, in this case, the students seem to deal so much with the individual vectors that parallelogram law simply did not occur to them.

Chapter 9 confirmed that the categorisation of students developed in chapter 4 was satisfactory and did not need have to be modified. However the interviews show the wide difference between the language used by students working at the lower and higher cognitive levels. The students working at the higher cognitive level use language, which suggest they deal with some kind cognitive unit. They may not be eloquent in the way they express it but the language they use is more powerful than the language which the students working at the lower stages use. The experimental lessons and reflection on actions were intended to move students to thinking about vector as a cognitive unit which in turn would allow them to be more flexible when using it.

10.6 Limitation of the study

The question arises as to whether the change is due to the teacher or the method. The study was done in one classroom in which I participated myself. It would be interesting to see if it could be repeated in another classroom with another teacher.

There were also practical limitations. Many students who would have been interesting to study further and could have given an interesting insight into some answers were not available for interviews. The groups were not well balanced as they started with different levels of the cognitive development. Three quarters of each group also studied Physics and it was difficult to assess the influence which the teaching of Physics had on the students' changes.

10.7 Directions for future research

As far as present research is concerned, the way of teaching students by focusing on the 'effect' of actions needs to be established in a school and tested with a wider range of students. If the premise is true that the use of non-verbal physical actions improves the students' sense of meaning, then, given the different views expressed by the teachers, it is important to discuss this aspect with them in a way that helps them too to gain an insight into the process. It is also important to discuss with them the language used in lessons and its meaning, to refine it and to improve the clarity of communication with all students at different stages of development. These suggestions should be the object of future research.

In general there is need for more research of the theory relating to embodiment and the symbolic compression. Some researchers (e.g. Pinto, 1998) have found that some students construct their ideas from their personal concept images while others do so from formal definitions and the structure of formal theorems. In the present research it was noted that some students, at the beginning of the course, were already at the highest stages of cognitive development of the concept of vector and had built the cognitive units themselves from the theory given in the earlier education. These students were successful without having any exposure to the embodied approach in the experimental Group. This suggests that, although an embodied approach may be useful to give overall statistical improvements in the class as a unit, there needs to be continued research into the needs of students who may think in different ways.

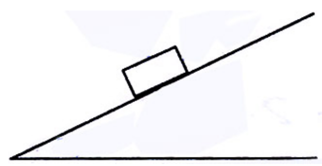
The notion of ‘effect’ of actions on base objects has applications in the construction of mathematical concepts encapsulated from processes. For example, the idea of two different actions having the same effect arises in a wide range of areas that are often interpreted in terms of an equivalence. For instance, equivalent fractions are different sharing procedures with the same effect, equivalent algebraic expressions are different procedures of evaluation with the same effect, and so on. A major line of research is to investigate the use of the focus on ‘effect’ in giving cognitive meaning to such mathematical concepts.

10.8 Reflecting on the effect of the study

In this thesis an approach was developed to make the transition from thinking of embodiments to manipulating symbols through the pivotal notion of *effect*. The results of the study revealed that there were significant changes in the class of students who followed this programme, in which they were more likely to conceive of the symbols for vectors as cognitive units that they could manipulate in a flexible and versatile manner. It is hoped that this fundamentally simple idea will be of use in improving the practical way in which teachers teach and students learn, not only in considering vectors but in every context where the effect of mathematical actions are represented by manipulable symbols.

Epilogue

Having completed this research I found it of value to return to the source of my original inspiration. The opening of this thesis referred to my increasing concern that students seemed to be able to learn to perform techniques to score highly on examinations, yet seem not to be able to apply their knowledge to slightly different situations, nor to retain their skills for ready use in subsequent courses. One particular question seemed to symbolize this problem that was used in the Preliminary Investigations, but did not feature in the main study. As the writing of this thesis came to a close, I decided to revisit the problem to see if my theoretical approach had made any long-term difference, not only to the concept of free vector, but also to the application of the ideas in other contexts such as mechanics.

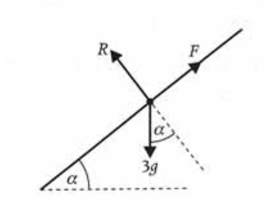


(draw the forces and resolve them parallel and perpendicular to the plane)

Fig. 10.1 Question on forces (a slope)

Most of our students could successfully resolve forces horizontally and vertically and solve problems using this technique, but they had serious problems in drawing the forces involved when a rectangular block was placed on an inclined plane (figure 10.1).

The analysis in this thesis suggests that students who approached vectors from an embodied viewpoint, focusing on the effect of translations to construct the notion of free vector in a meaningful way, would be able to build a mathematical concept that they could use in other contexts. I decided to give a variant of the original problem to several different groups of students, a year and a half after the students involved in the research had finished their course on vectors and half a year after their exams (figure 10.2).



A particle of mass 3 kg slides down a rough plane at an angle α = 30° to the horizontal. If $\mu = 0.5$ find the acceleration of the mass.

Fig. 10.2 Revisiting the original problem

The question was given to four groups in all. Two were the groups who had been involved in the Main Study:

> ***Group A** who had been given the experimental treatment in Year 12 and were now at the end of 13,*
>
> ***Group B** who had been given the standard treatment in Year 12, but subsequently had revised the work with me in Year 13 including two plenary sessions.*

Two other Groups were also included from Year 12:

> ***Group C**, in year 12 who were taught by a teacher who had been interviewed as part of the research and had shown interest in the ideas I had used and had adopted the techniques in her own teaching it.*
>
> ***Group D,** in Year 12, taught by a teacher who was not involved with the research.*

Every student in Group A answered correctly**. **All students in Group B except three answered correctly**. The three who answered incorrectly made the error of omitting the parallel component of weight (as in happened in the Preliminary Investigations). When I checked my register, I realised that **all three who made errors were absent for the experimental revision lessons.

In Group C**, taught by a teacher aware of the experimental technique, **four out of six students answered correctly** while two missed the parallel component of weight in their calculations. This supports the idea that the method may be used successfully by other teachers. However, **in Group D, only one out of eleven students answered

correctly *while the rest of them missed the parallel component of weight in their calculations.*

The data is gratifying. It shows that the class of students who were taught in the standard way continued to have difficulty with the resolution of forces with only one out of the whole class responding correctly. Meanwhile, almost all of the students who used the focus on effect, even for a short time, conceptualized the forces correctly several months after the lessons were given. They had not only conceptualized the idea flexibly, they had retained the ideas after the passage of time.