THE RELATIONSHIP BETWEEN PHYSICAL EMBODIMENT AND MATHEMATICAL SYMBOLISM: THE CONCEPT OF VECTOR

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This article considers the relationship between the perceptual world of embodiment and the conceptual world of mathematical symbolism with particular reference to the concept of vector. We show that the embodied world has a variety of different meanings dependent on the context in which the concept occurs and consider the way in which these are linked to mathematical symbolism. The learning of vectors is situated in the intersection of embodied theory relating to physical phenomena, and process-object encapsulation of actions as mathematical concepts. We consider the subtle effects of different contexts such as vector as displacement or force, and focus on the need to create a concept of vector that has greater flexibility. Our approach refocuses the development from 'action to process' as a shift of attention from 'action to effect' in a way that we hypothesise is more meaningful to students. At a general level we embed this development from enactive action to mental concept within a broad theoretical perspective and at the specific level of vector we report experimental data from a first study of the application of the theory.

INTRODUCTION

This paper has a dual focus. It introduces a general view of mathematical development that highlights three distinct modes of operation in mathematics and it applies this theory to the specific case of the concept of vector. It follows the natural development of the individual, building from physical interactions with the world, developing increasingly sophisticated meanings for real-world phenomena; at the same time, mathematical symbolism is developed to allow greater accuracy of computation and greater precision in solving problems. For the minority of students who go on to study mathematics at university, more formal mathematical theories are developed based on formal definitions and logical proof. Our analysis yields three distinct mathematical worlds, each with its own way of constructing concepts and with its own way of proof. These are respectively, embodied, symbolic and formal. We will consider each of these in detail later. For the moment it is sufficient to know that the embodied world begins with our interactions with the physical world through our senses, but, through introspection we gradually develop more sophisticated ideas that retain their links to perceptual senses but no longer have physical essence, such as straight lines that have no width but can be extended arbitrarily in each direction. The symbolic world that we speak of focuses on the symbols typical of arithmetic and algebra, which Gray & Tall (1994) noted have dual roles as process (eg addition) and concept (eg sum), and gave the name 'procept'. The third world is the formal world of axiomatic systems built from axioms, definitions and formal proof. The three worlds of which we speak are therefore called *embodied*, *proceptual* and *formal*.

Vectors have distinct aspects that are representative of these three worlds. A vector is embodied in physics and mechanics in a variety of ways: as a force, a transformation, a velocity, and acceleration, or, more generally as a quantity having magnitude and direction. Addition of vectors is performed by placing them nose to tail and finding the vector with the same cumulative effect. Vectors as procepts begin with the idea of a translation (in the plane or three or more dimensions) and are represented symbolically as column matrices, with addition performed by adding the components.

The formal concept of vector is defined in terms of a vector space which consists of a set of elements (with no specific meaning except that they are called vectors) acted upon by a field of scalars satisfying certain properties. Formal addition of vectors is performed by using an axiom that asserts that for any two vectors **u**, **v** there is a third vector **u+v** which satisfies other properties, such as **u+v = v+u**.

We shall discuss these aspects in greater depth in the paper, focusing mainly on the two worlds of mathematics that are met in school—the embodied world and the symbolic world of procepts. For the present, it is sufficient to notice that vectors are conceived in quite different ways in each world (and in the embodied world, there are several distinct contexts of meaning), there are different ways of performing constructions (such as addition and scalar multiplication) and there are different forms of proof. Our concern is how students might be introduced to this rich yet complex mathematical structure in a flexible way that enables them to use the various aspects in a powerful and effective manner. As we will shortly see, the early educational research on vectors has been performed mainly in the context of science education, with a focus on the modelling of physical phenomena by mathematical symbolism. This is consistent with the curriculum being used in England where the student is introduced to vectors initially in a physical context and the ideas are then intended to give rise to the corresponding mathematical ideas.

We believe that such an approach has additional complexities that arise from the range of physical experiences that give very different sensory meanings. For instance, a vector as a transformation 'feels different' from a vector as a force or as a velocity. Thus the embodiments of the concept of vector lead to a range of conscious and unconscious beliefs that can cause obstacles to drawing together the various aspects into a central core mathematical concept. For example, we will show how the embodiment of vector as a journey lead more naturally to the use of the triangle law and vector as a force leads more naturally to the parallelogram law; for many beginning students, this can lead to significant misconceptions in mechanics. Thus embodiment has its special peculiarities that require attention.

The mathematical concept of vector also has a subtle range of meaning, which, in the secondary school, inhabits the embodied world and the proceptual world. It may be introduced as a translation of a geometric object in the plane. This has a dynamic aspect as a process, which may also be represented by an arrow that embodies both the feeling of dynamic movement and the concept of the arrow as an object. Thus the vector as procept may be given an embodied object conception as an arrow. Likewise the symbol $\begin{pmatrix} x \\ y \end{pmatrix}$ carries both the sense of a process as a shift with components *x*, *y* and a concept as a vector quantity, having the dual meaning of process and concept inherent in symbols as procepts.

Our theory therefore has aspects of embodiment (as represented in the works of Lakoff and his colleagues, Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) and of the process-object theories of Dubinsky (1991), Sfard (1991), and Gray & Tall's theory of procepts (Gray & Tall, 1994, Tall *et al*, 2001). Later in the paper we will discuss how it relates to a wide range of other developmental theories such as those of Piaget, Bruner and the SOLO taxonomy of Biggs and Collis (1982, 1991).

Our theoretical perspective, focusing on embodiment and mathematical symbolism, offers ways of linking the communities of practice of physicists (with an emphasis on the physical meanings and their translation to mathematical equivalents) and mathematicians (with a focus more on the core mathematical concept, through symbolism and later to formal proof). A student taught in these two communities must learn to build a flexible mental structure that can appreciate both viewpoints and yet make a bridge between them. Our view is that this should be fostered by reflective plenaries in which the teacher, as mentor, encourages students to reflect on the range of ideas, to focus on the essential core ideas and to practice communicating these ideas to others (Watson, 2002). The experimental evidence presented in this paper therefore has three fundamental aspects:

- an analysis of conceptual differences between embodied conceptions of vector, in particular the use of the triangle and parallelogram law,
- a study which compares the effects of the use of reflective plenaries in an experimental group compared with a control group using traditional teaching methods.
- a study that compares the effect of an embodied foundation for vectors as translations of figures in the plane with a traditional approach introducing vectors as quantities having given magnitude and direction,

In the next section, we review published research into the concept of vector and consider its effectiveness. This leads to a discussion of the conceptual differences in the notion of vector as a force and vector as a translation with direction and magnitude and our empirical evidence distinguishing conceptual differences between embodiments of forces and translations. We then discuss the general theory of the three worlds of mathematics and their differing modes of operation, applying this to the case of vector. This leads us to the need for designing a mechanism whereby students can make sense of the rich complexity of ideas, based on their physical interaction with the world and the connections with the use of mathematical symbolism. We then consider empirical evidence from our studies which helps us to build and test our theoretical approach.

RESEARCH FROM SCIENCE EDUCATION

Earlier research into vectors and mechanics began in Science Education:

> The great wealth of information and comment about students' understanding of mechanics concepts has been produced virtually exclusively by science educators. This seemed to suggest that it was all the more important that teachers of mathematics should be made more aware of the difficulties their students were experiencing in learning mechanics. Orton (1985. p.8)

Mechanics is seen to incorporate many different concepts and demands various abilities of understanding by students. These include knowledge of concepts in physics and from mathematics, including trigonometry and vector calculus. Freudenthal argued:

> Among the sciences, mechanics is the closest to mathematics, in particular to geometry.
> Freudenthal (1993, p 72).

His view is echoed by Crighton:

> Of all the fields to which one might wish to apply mathematics, mechanics has the strongest claim to a very prominent place in syllabuses for mathematics in U.K schools and universities. Crighton (1985, p 10)

Freudenthal saw mechanics as a key area for introducing mathematical thinking into real-world situations:

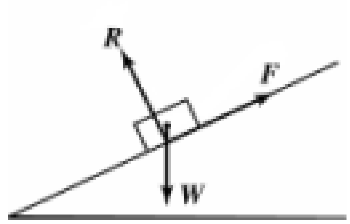
> A key concept in nowaday's views on mathematical invention is *mathematising*, which means, turning a non mathematical matter into mathematics, or a mathematically underdeveloped matter into more distinct mathematics Freudenthal, (ibid, 72).

Kitchen, Savage & William (1997, p 165) advocated that mechanics "should be seen not only as the natural partner of Pure Mathematics but also an essential grounding for understanding modelling". This modelling however, is more than a direct link from the physical world to a mathematical framework. Between these it is necessary to have a "scientific" model of a physical situation that ignores less important aspects and focuses on the essential matters that can be modelled in a mathematical manner:

> It is to be noted that the interaction involving the real world is always with the scientific model, and that involving mathematics is between the mathematical model and the scientific model. The term "scientific model" refers here to a comprehensive set of idealizations and abstractions from the real world, comprising well-defined quantities amenable to experimental measurement and comprising also the basic laws relating this quantities. In classical mechanics, for example, the scientific model comprises the notions of mass, momentum, energy, angular momentum […] and the basic laws are simply Newton's laws, supplemented with other general laws where appropriate.
> Crighton (1985, 11)

For example, if we consider a brick on a sloping plane in the real world, then the scientific model idealizes the situation and sets up a system with three forces, the weight W due to gravity, the normal reaction R and the frictional force F. Resolving this model perpendicular and parallel to the plane, and taking account of the relationship between friction and normal reaction gives a set of mathematical equations to solve for a particular piece of information, for instance, the maximum angle θ before the brick begins to slide (figure 1).

This leads to the standard cycle of modelling real world problems in mathematics:



$$F = W \sin\theta$$
$$R = W \cos\theta$$
$$F \le \mu R$$

Real world | Scientific model | Mathematical model

Fig 1: The translation between a real world situation and a mathematical model

> … we make a problem set in the real world and first formulate it as a mathematical problem; this together with any assumptions made is the mathematical model. The mathematical problem is then solved and finally the solution is translated back into the original context so that the results produced by the model can be interpreted and used to help the real problem. Berry & Houston, (1995, p.1)

The initial translation of the real-world problem into a conceptual idealization may make gross simplifications of the original to be able to formulate a solvable mathematical model, as in the following excerpt:

> Mechanics is concerned with the motion of objects and in many cases is rather complicated. Take, for example, the motion of a leaf falling to the ground.
>
> To start such problem we simplify it, considering first the motion along straight line. We ignore any rotation and follow the path of some point in or on the object. This is very important simplification in mechanics. A point object is called a particle. It is a mathematical idealization of an object in which all the matter of the object is assumed to occupy a single point in space. The particle has a mass of the object but no dimension.
>
> Berry, Graham, Holland & Porkess (1998, p 10)

In this particular example (where the leaf will also be affected by air pressure, and is likely not to fall down in a straight line), the actual motion may have aspects that are visible, but must be ignored in the modelling process that must concentrate on a simplified aspect: motion due to gravity. Sensory experiences may operate in subtle ways that make this modelling process far more subtle than is apparent on the surface.

Force

The concept of force arises in many ways, including physical muscle experiences related to our actions on the 'world of life' (Freudenthal, 1993, p. 73). It is one of a variety of concepts in physics with a range of sensory meanings:

> As such let me mention frequency, speed, even acceleration, but also density (which I will recommend as an access towards mass). What regards force, one has, in the first instance, to contend with the everyday semantic troubles, which are a well-known linguistic feature; but in the language of physics the meaning of "force" has been settled only after centuries long hesitations, and even now such forces as the centrifigural one are kept alive, albeit with the adjective "apparent".
>
> Freudenthal (1993, p 74)

It has encountered many difficulties in the history of sciences. It is well known that the main obstacle of non-understanding by Aristotle of inertia lasted into the middle ages. For the student the notion of force arises from its corresponding physical sensations including experience of the forces of actions and reactions on the environment which we recognize as weight, friction, pull, push etc. It is an *embodied concept* (Lakoff & Johnson, 1999) involving many different bodily sensations.

These physical sensations are often suggested as a basis for building the concept of vector. Aguirre and Erickson (1984) studied the understanding of vector from a physical point of view, focussing on the ideas of students in which the mathematical concepts entangle with the physical concepts. They suggest that

> … teachers could [...] built upon students intuitions (developed through experience in everyday settings) by relating these intuitions to the more formal problem settings in the scientific domain. Aguirre & Erickson, (1984, p 440)

We agree with this viewpoint and propose to build our approach to vectors on physical activities. However, the notion of force continues to involve a wide range of misconceptions by students and teachers documented in the science education literature (Williams, 1985; Brown, 1989; Kruger, Palacio & Summers, 1992; Thijs, 1992; Bar, Zinn, Goldmuntz & Sneider, 1994; Palmer & Flanagan 1997; Shymansky *et al*. 1997). After more than a decade of such developments, Rowlands, Graham and Berry conclude:

> … various attempts at classifying student conceptions have been by and large unsuccessful [...]. A taxonomy of students conceptions may be impossible because the considerations of 'misconceptions' require a specific regard for the framework from which the 'misconception' occur [...] and how misconception is linked to the other forms of reasoning. Rowlands et al, (1999, p 247)

Subtle distinctions in embodied concepts.

Our preliminary experimental evidence emphasizes the subtle nature of the embodied concept of vector. We find that students' perceptions of forces acting on their own bodies cause them to sense the combination of two forces as a single force in between. Pulling forward on two arms causes the individual to move in a forward direction representing the combination of the two forces. This gives rise to a natural interpretation of vector addition by using the parallelogram law. On the other hand, vectors met in the context of successive transformations leads more naturally to the combination of vectors using the triangle law. As far as the usual analysis of the mathematics is concerned, these two laws are, of course, mathematically equivalent. However, we found that they are not *cognitively* in agreement for the student meeting the ideas for the first time. The subtle differences prove to lead to errors in modelling physical phenomena.

For instance, in a preliminary experiment, we found that students could be very efficient at learning to resolve forces horizontally and vertically. For example, in a written test, 25 out of 26 students in a class could respond correctly to the problem given in figure 2.

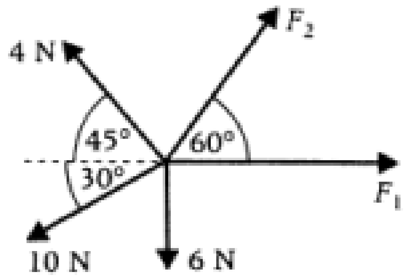


Fig 2: find F_1, F_2

However, when the same students were given the problem in figure 3, only 4 out of 26 were successful. Thus almost all were successful when resolving horizontally and vertically and almost all were unsuccessful when specifying the forces on the block and resolving perpendicular and parallel to the plane.

A naïve analysis of their difficulties might suggest they had difficulties when the resolutions were required at an angle rather than horizontal and vertical. In interviews, however, it transpired that a more subtle underlying effect was operating.

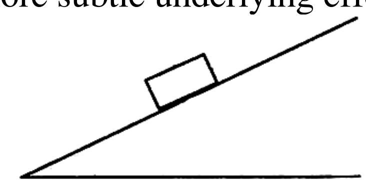


Fig 3: describe & mark forces

The students' experience had been built from the notion of vector as a journey or translation. In addition, during their recent work on forces, they had spent considerable time focusing on the resolution of forces into their components, which had been done using right angled triangles. Consequently, their natural method of adding vectors was through the triangle law. When resolving forces into horizontal and vertical components, the diagram uses triangles that reinforced their use of the triangle law and the triangle law came to dominate their interpretation of vector addition. Figure 4 shows a typical use of a triangle of forces to picture the components of the vertical force due to gravity.

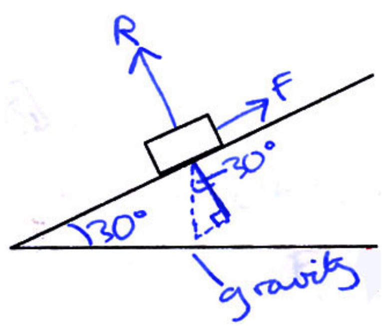


Fig 4: forces as marked

The student has marked the vertical line of action of gravity and resolved it using the triangle law. He considered the component parallel to the plane was acting along the lower side of the triangle. As this is far away from the block itself, he did not sense this component acted on it and omitted it from the forces acting parallel to the plane.

The physical embodiment of the vector concept therefore has highly subtle effects on students' conceptions. To build a bigger picture, we place this observation in a broader theoretical framework.

Theoretical Framework

The case of vector is typical of the cognitive developments of mathematical concepts, reflected in a wide range of theories of development. Arithmetic begins in the embodied world of sorting and counting physical objects, developing symbolism to represent the operations more efficiently, and, at a formal level, to express the properties in terms of axioms specifying properties such as commutativity, associativity and the distributive law for addition and multiplication. Fractions likewise begin with physical operations of sharing that are represented by the symbolism of fractions and the ultimate formal theory of rational numbers.

Piaget's theory of development through stages begins with the sensori-motor stage, passing through a pre-conceptual stage to first a concrete-operational mode, then a formal-operational mode. Bruner begins his theory of modes of representation through enactive, then iconic, then symbolic. In this case, his symbols encompass not only language, but also the mathematics symbols of arithmetic and the formal symbols of logic. Biggs and Collis (1982), in their analysis of student responses (the SOLO taxonomy measuring the Structure of Observed Learning Outcomes) combine aspects of Piaget and Bruner to formulate their own sequence of modes of operation as

sensori-motor / ikonic / concrete symbolic / formal / post formal.

This taxonomy explicitly notes that, as each new mode comes on stream, it is *added* to the previous available modes rather than replacing them. This accumulation of modes of operation is fundamental to our theory. However, the SOLO taxonomy is not designed as a developmental theory, it focuses mainly on measuring the outcomes of learning by classifying responses to student assessment. In addition to the succession of modes, it proposes a cycle of learning in each mode: unistructural (responding in terms of a single element), multi-structural (responding with several unconnected elements), relational (putting these elements together), extended abstract (having a global grasp of the whole structure). We too see such cycles of learning in operation, however, in line with Pegg (2002), we see such cycles applying not to a mode as a whole, but as a developmental cycle for each conceptual structure that is met by the student. There is, generally, an initial exploration, focusing on separate elements of a situation, a putting together of the ideas and the relationships between them, leading to a global view of the structure, perhaps encapsulating it as a total entity. In this way we see the global theory of modes of operation being complemented by local cycles of concept learning.

In considering different types of mathematical concept, Gray & Tall (2001) wrote:

> For several years […] we have been homing in on three[…] distinct types of concept in mathematics. One is the embodied object, as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. Another is the symbolic procept which acts seamlessly to switch from a "mental concept to manipulate" to an often unconscious "process to carry out" using an appropriate cognitive algorithm. The third is an

> axiomatic object in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory. (Gray & Tall, 2001, p.70)

We take this a step further. We suggest that each of these distinct types of concept is handled in a different way, with differing kinds of operations and differing kinds of belief in the nature of truth. We therefore refer to the distinct modes of operation as three distinct worlds of mathematics:

i) The *embodied world* of perception and action, including reflection on perception and action, which develops into a more sophisticated Platonic framework.
ii) The *proceptual world* of symbols, such as those in arithmetic, algebra and calculus that act as both *pro*cesses to do (e.g. 4+3 as a process of addition) and concepts to think above (e.g. 3+3 as the concept of sum.
iii) The *formal world* of definitions and proof leading to the construction of axiomatic theories.

These worlds develop in sequence, first the embodied world in a sensori-motor form, that has its roots in the sensori-motor/ikonic modes of thought based on physical perception and action. Then the proceptual world builds from embodied actions such as counting, adding, grouping and sharing, which are given symbolic forms for number, sum, product, division, and so on. As the child grows, both these worlds are available. The embodied world gives physical manifestations of concepts such the number of objects in a collection, the number track (consisting of discrete entities placed one after another), the number line (consisting of a continuous measure), the plane containing its geometric figures and the actual world of three-dimensional experience. The proceptual world develops ever more sophisticated structures: whole number arithmetic, fractions, decimals, algebraic expressions, functions of various kinds. At all stages links occur between the two, with the embodied world being sensed perceptually and reflected upon to produce more sophisticated mental conceptions which, at every stage, correspond to symbolic structures such as the number system, coordinates, symbolic graphs, algebraic surfaces.

Much later, formal theories develop, based not on existing objects or actions, but on formally defined axioms and definitions which prescribe properties of axiomatic systems such as groups, rings, fields, vector spaces, metric spaces, topological spaces, and so on. When these definitions are formulated, there are already many examples of each which suggest properties that the structures are likely to have. However, all properties other than those assumed in the axioms and definitions, must be deduced logically from the axioms and definitions in a sequence of theorems, which themselves can be put together to deduce new theorems.

Language has a vital role in focusing at an appropriate level of structure in each of the worlds of mathematics

. Metaphorically it acts as a magnifying glass, allowing us to focus on detail, or to stand back and focus on the overall structure, as Skemp (1979) has theorised in his *varifocal* learning theory. We envision the embodied world growing in sophistication in a way that is consistent with Van Hiele's (1986) theory of geometrical growth. Real world objects are seen, manipulated and sensed in various ways and the perceptions are refined through experience and communication with others. Objects are observed to have increasingly subtle properties, which enable categories to be first described by their properties, and then defined by their properties. Conception of the external world grows through reflection on internal conceptions of embodied experiences.

The proceptual world builds through a range of subtly different process-object relationships in the arithmetic of whole numbers, fractions, decimals, irrationals, reals, in algebra and then in the potentially infinite limit processes of calculus. (Tall *et al*., 2001). The properties and relationships observed in the objects of the embodied world and the procepts of the proceptual world form a basis for the shift to axiomatic definition and proof characteristic of the formal world (Tall, 1995). Indeed, each of the three worlds focuses on essentially different qualities, the embodied world focuses on *objects* and their properties, the proceptual world on *processes* represented by symbols, the formal world on *properties* and deductive relationships between them.

Validity in different worlds of mathematics

Proof, which is the central idea of the formal world, has earlier manifestations in the embodied and proceptual world, each of which has a distinct notion of validity. In the embodied world, validation is through prediction and experiment. Things are true in the embodied world initially because events occur in a predictable way. We can *see* that two and three make the same total as three and two by holding up the requisite number of elements and the total is the same irrespective of the order. This 'order irrelevancy principle', later called 'the commutative law', arises through experience and observation. In the embodied world it is *not* a 'law' imposed *on* the real world, it is an *observation* of what happens *in* the real world.

In the proceptual world, truth can be tested by computation and manipulation. There are some 'truths' such as '$x+y=y+x$' or '$x(y+z)=xy+xz$' which we 'know' to be true from our experience of arithmetic and assume that it will naturally hold in algebra. There are other truths, such as

$$(a-b)(a+b)=a^2-b^2$$

that we can show to be true by carrying out simpler operations of algebra that we already 'know' to be true. For instance:

$$(a-b)(a+b)=(a-b)a+(a-b)b=a^2-ba+ab-b^2=a^2-b^2.$$

Those who succeed in the proceptual world become accustomed to carrying out these procedures, relying on mental imagery and on the guidance and approval of a teacher to give a coherent manipulative facility consistent with the community of practice of mathematicians.

In the formal world, the situation changes again. Statements like '$x+y=y+x$' or '$x(y+z)=xy+xz$' are no longer true because of one's prior experience, they are true because *they are assumed to be true as axioms* for an axiomatic mathematical structure, say in a field or ring. The symbol $x+y$ need no longer carry with it any meaning of addition or involve any actual process of computation. What matters is that the structure obeys the given axioms related to the symbols and then, having proved a result solely by logical deduction from the axioms, one knows that this result is then true of *all* the examples in which the axioms are satisfied.

The case of vector

As an example of this theory, we consider the case of a vector. This begins in the embodied world as a quantity with magnitude and direction, having a variety of manifestations as journey, transformation, velocity, acceleration, force, momentum, and so on. We have already seen how different manifestations, such as force or transformation, can offer different meanings that can then lead to difficulties in moving from these embodiments to a symbolic equivalent. First there is the move from the real world context to a scientific model, say as an arrow having magnitude and direction. This physical model still lies in our embodied mode of thought, now however, invested with a greater precision of meaning. We can *see* a vector as an arrow with magnitude and direction; we can *add* two vectors together by shifting the second (preserving its magnitude and direction) so that its tail is on the nose of the first and defining the sum as the arrow given from the tail of the first to the nose of the second (figure 5).

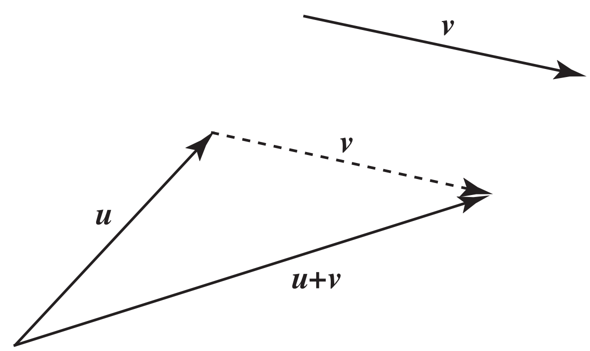


Figure 5: the vector sum of *u* and *v*

If we were to ask how we 'know' that **u+v** is the same as **v+u**, the answer lies in the physical embodiment that **u**, **v** are sides of a parallelogram. This 'knowing' arises in the embodied world either as a sense of the properties of a parallelogram or as part of a more sophisticated result of Euclidean proof.

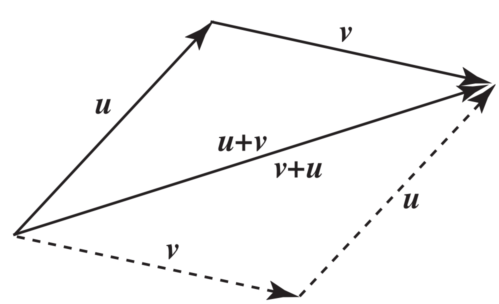


Figure 6: u+v is the same as v+u

Although these ideas are evident from a sophisticated viewpoint, there are problems that may affect the learner. For instance, if one 'feels' that a vector is like a force acting at a point, say a push on your left shoulder, then that force has a very different effect from a force of equal magnitude and direction on your right shoulder. To give a sound embodied sense of these ideas we theorised that we would need to give students appropriate embodied activities that are the focus of later discussion in this paper.

A vector may be represented symbolically by a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. This symbol is a procept; it carries with it the meanings of both process (of a shift by components *x*, *y*) and of a concept (the column vector that can be treated as an entity and operated upon). For instance the sum of two vectors can be calculated precisely by adding the components. The addition of two vectors is now seen to be commutative because the addition of each component is commutative.

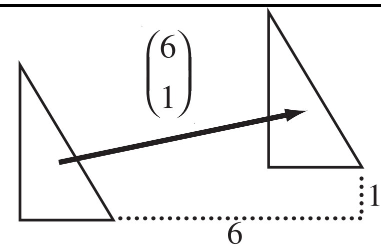
In the formal world, a vector is no longer a quantity with magnitude and direction. A vector is any element of a set *V*, which has defined an operation of addition and multiplication by an element from a field *F*, satisfying a list of axioms. The sum of two vectors is now part of the definition and the commutativity of addition is one of the axioms. The properties found in the embodied and proceptual worlds of vectors are now turned around to become the foundation of the axiomatic theory. The focus is now on the list of axioms and what can be deduced from those axioms. For instance, one might define a *linear sum* of vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$ to be a sum $\lambda_1\mathbf{v}_1+\ldots+\lambda_n\mathbf{v}_n$ where $\lambda_1,\ldots,\lambda_n$ are scalars. One might say that a set $\mathbf{v}_1,\ldots,\mathbf{v}_n$ of vectors is a *spanning set* of *V* if every vector in *V* is a linear sum of these vectors. A set $\mathbf{v}_1,\ldots,\mathbf{v}_n$ is said to be *linearly independent* if the only way that $\lambda_1\mathbf{v}_1+\ldots+\lambda_n\mathbf{v}_n$ can be 0 is when $\lambda_1=\ldots=\lambda_n=0$. Finally a *basis* (if it exists) is a set of vectors that is both a spanning set and linearly independent. The big theorem is that if a vector space has a basis with *n* elements, then any other base has the same number of elements. It is then called a *vector space of dimension n*. In a vector space of dimension *n*, every vector is uniquely of the form $\lambda_1\mathbf{v}_1+\ldots+\lambda_n\mathbf{v}_n$ and the coordinates $(\lambda_1,\ldots,\lambda_n)$ uniquely specify the vector. Writing the coordinates as a column matrix $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ reveals the formal idea of a vector in a vector space (of dimension *n*) to be nothing more or less than the proceptual idea of a vector, but with one vital difference. We now *know* that we can prove all the properties of vectors from the list of axioms of a vector space, to give a coherent and consistent formal theory.

In school mathematics, of course, we do not travel as far as the formal theory. We do, however, deal with embodied and proceptual notions in two and three dimensions. We therefore have to deal with the various embodied meanings (and the possible difficulties of the kind presented in the previous section) and relate these to the geometric and symbolic notions of vector. Embodied theory (Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) is appropriate for the initial sensori-motor and perceptual experiences on which the notion of vector is built. Lakoff claims that by metaphorical thought these embodied experiences lead naturally to the more mathematical meanings. Our perceptions of the growth of the mathematical theory are supported not only by embodied meanings, but also through the flexible use of symbolism as process and concept (Tall *et al*. 2001), itself very much influenced by the idea of encapsulating (or reifying) a process as a thinkable object (Dubinsky, 1991, Sfard 1991). The latter theories begin with the idea of carrying out some kind of action on already known objects (eg a vector as the action of translating a shape on a plane), which is then interiorized as a process (where what matters more is the *result* of the process, rather than the specific steps made in a particular act) then encapsulated as a thinkable mental object.

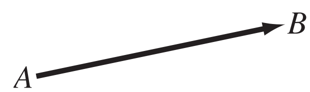
ANALYSIS OF THE SCHOOL APPROACH TO VECTORS

The text-book by Pledger *et al*. (1996), used in the school for introducing vectors to the students in the year prior to our study, follows a pattern that is reminiscent of the encapsulation of processes as objects. In this approach, the processes are translations of objects in the plane and these lead to vector concepts, as follows:

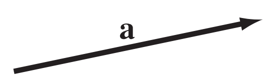
1. translations are described using column vectors, $\begin{pmatrix} x \\ y \end{pmatrix}$ […] with the column vector $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ meaning 6 units in the positive x direction and 1 unit in the positive y direction.



2. an alternative notation which can be used to describe the translation is $\overrightarrow{AB}$ representing where A is the starting point and B is the finishing point. […] The lines with arrows are called **directed line segments** and show a unique **length** and **direction**



3. a third way to way to describe a translation is to use single letters such as **a** …. Translations are referred to simply as **vectors**. […] [Each vector] has a unique **length** and **direction** ...



4. **Position Vectors**. The column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ denotes a translation. There are an infinite number of points which are related by such a translation. … The diagram shows several pairs of points linked by the same vector. The vector which translates O to P, $\overrightarrow{OP}$, is a special vector, the *position vector of P.*

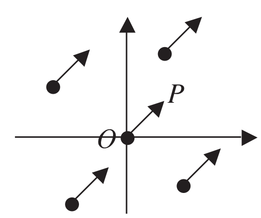


Figure 7: A development of vector concepts

This pragmatic approach has some of the aspects of process-object encapsulation. Stage 1 begins in the physical world of embodiment with a vector as an action on a physical object. The translation may be represented as an arrow. In stage 2 the object is omitted and the focus of attention is on the line segment as a journey from a point A to a point B. Stage 3 shifts the focus to the vector as a single entity drawn as an arrow and labelled with the single symbol **a**. This entity has both an enactive aspect (the movement from tail to nose of the arrow) and an embodied aspect (the arrow itself).

The compression from the notation $\overrightarrow{AB}$ to the single symbol **a** has certain simplifying features. For example if $\overrightarrow{BC}$ is a second vector denoted by **b**, the commutativity property

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$$

is expressed far more naturally in this form than in the form

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}.$$

A student who treats a vector at an early stage as a journey can conceive of the left hand side moving along AB then along BC. But what does the right hand side mean? How can one go along BC and then along AB without a jump in the middle? As journeys, the order of taking the journey matters! To give meaning to this idea requires the full use of equivalent vectors.

In stage 4 the column vector is used to denote an infinite family of arrows with the same length and direction, with one specific vector starting at the origin singled out as a position vector as a special representative of the whole family. (This lays a possible foundation of later formal set-theoretical ideas, in terms of equivalence relations and equivalence classes; that is not considered here.)

A distinct weakness of this curriculum is that the students involved have not had any experience of Euclidean constructions. Thus, for example, they cannot add two vectors accurately by a Euclidean method that moves a second vector to fit on the end of a first vector by constructing a line parallel to a given line through a given point.

In practice, the physics teachers prefer to 'simplify' the ideas by referring separately to horizontal and vertical components of vectors. For example, to add two vectors, they would consider each vector separately, calculate its horizontal and vertical components and add them together to get the components of the sum. Some of the exercises were performed on squared paper so that the calculation of the components could be performed by counting squares. In parallel, the students would often use the equivalent matrix method to add vectors in pure mathematics. Thus, although they had been taken through the spectrum of development in stages 1 to 4, to make any computations, they usually fell back to level 1.

ACTIONS AND EFFECTS – THE INSIGHT OF A SPECIAL STUDENT

In attempting to build a more flexible conception of the notion of vector that encapsulates the whole structure of embodiment and process-object encapsulation, it is of value here to move ahead and consider an incident, which occurred in the first study that will be described shortly. We were struck by the interpretation formulated by one particular student whom we will call Joshua. He explained that different actions can have the same '*effect*'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors.

Joshua: You can add vectors to get the resultant [he draws two vectors separately then draws them both again approximately parallel and ‘nose to tail’ and shows the resultant using the triangle method of addition].

Teacher: What does it represent?

Joshua: It is like having two effects being represented by one effect. To find the total effect of two is like adding two separate effects. We can add more vectors but they would not make a triangle but a polygon.

By focusing on the effect, rather than the specific actions involved, we realised that it proves possible to get to the heart of several highly sophisticated concepts. For instance, in fractions, ‘divide into three equal parts and take two’ is a different action from ‘divide into six equal parts and take four’ but they have the same *effect, g*iving rise to the central idea of equivalent fractions. The same idea occurs in algebra where $2(x + 4)$ and $2x + 8$ involve different sequences of actions with the same effect, leading to the notion of equivalent expressions. We hypothesize that the notion of action-effect is a more approachable way of describing the theory of action-process (Dubinsky, 1991) or procedure-process (Gray & Tall, 1994).

Figure 8 shows a hand placed on a triangle, shifting it by a translation on a plane. Any arrow drawn from, say the starting point of a particular finger to the finishing point of the same finger represents the movement of that point on the triangle. All such arrows have the same magnitude and direction. Any of these arrows represents the magnitude and direction of the translation. They all have the same *effect*.

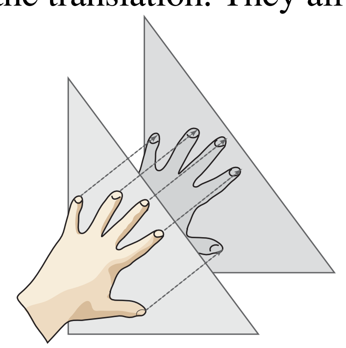


Figure 8: A translation of a figure on the plane

We may take this idea well beyond that which arose with Joshua. Instead of dealing only with straight-line translations, we note that it does not matter *how* the translation occurs, all that matters is the relationship between the initial and final position. So, if the movement wobbled on the journey (as in figure 9), the effect of the translation interpreted in terms of the initial and final position is the same as if it had been moved in a straight line (figure 8). What matters is not the *procedure* of movement from one place to the other, but the *effect* of that procedure.

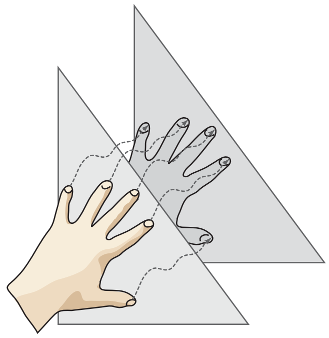


Figure 9 : another movement having the same effect

This has an interesting consequence in terms of conceptualising the notion of addition of vectors. The effect of a shift from A to B, then B to C is the same as a shift direct from A to C. *The sum of two free vectors is simply the single free vector that has the same effect* (figure 10).

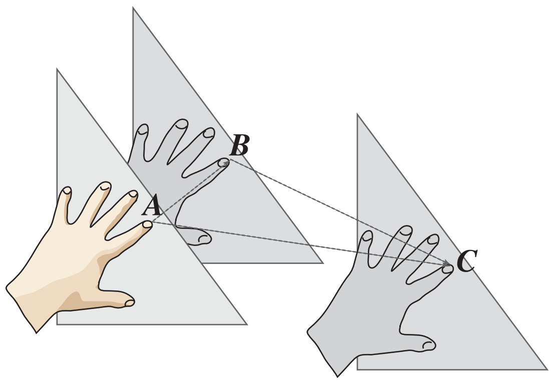


Fig 10: The sum of two vectors as the total effect of two translations

By performing translations physically and focusing on their effect, we hypothesised that we could help students build a notion of a translation as 'free vector' in such a way that they appreciated the equivalence of free vectors with the same magnitude and direction and the flexible use of equivalent vectors for vector addition.

We believe the notion of 'effect' to be an important stepping-stone in cognitive development that links to concepts in all three worlds of mathematics. In the embodied world, it enables a refocusing of thought from a process to the *effect* of that process. In the proceptual world it corresponds to the cognitive compression of mental processes into thinkable objects in which processes—such as addition, multiplication, sharing—become concepts—such as sum, product, fraction—in which the symbol allows the thinker to pivot between one meaning and the other. In the formal world, using a set-theoretic approach, it corresponds to the notion of equivalence relation.

In the 'New Mathematics' of the 1960s, there was an attempt to introduce an approach using equivalence classes in set-theory at school level. For instance, to give a formal meaning for the fraction m/n, one introduced the set P of pairs of integers of the form (m,n) then defined two pairs (m_1,n_1), (m_2,n_2) to be equivalent if $m_1n_2 = m_2n_1$. A rational number is then an equivalence class of ordered pairs, such as $\{(3,4), (6,8), (9,12), \ldots\}$. This formal approach, even when expressed informally, failed to become established. Although it seemed a natural way for experts to organise the curriculum, it did not work for the cognitive development of students. Our approach occurs in the embodied world, not the formal world. Here the notion of 'same effect' has not only a greater chance of being understood, it also lays down links that have the potential to translate into the notion of equivalent symbols in the proceptual world and equivalence classes in the formal world.

Essentially our teaching is aimed at students giving meaning to concepts in the embodied world, and then sharing their experiences with a mentor who guides them to express their ideas to each other in ways that embody ideas with proceptual and (at a later stage) formal equivalents. This approach does not 'talk down' formal theory in the failed manner of 'New Mathematics'; it starts from the students embodied experiences to help them construct the core mathematical ideas in a meaningful way.

EMPIRICAL STUDY

Use of Plenaries in English Schools

In English Education there has been a concerted effort to encourage students to develop efficiency in calculation complemented by conceptual understanding of fundamental ideas. In the early years (aged 5–13), the Numeracy Strategy specifies that all classes should normally consist of three stages: starter, core, and plenary. The starter activity, with the teacher relating to the whole class, sets the scene for the main core of the lesson, in which students work in groups or on individual tasks, and the final plenary which either looks forward to the next stage of development in the next lesson, or reflects on the ideas met in the lesson and makes connections between them.

In this research, we follow a pattern of an opening starter session to focus students on the ideas to be explored in the core part of the lesson, with a plenary to reflect on the core and to connect the ideas together in a focused way.

Two investigative studies

So far we have conducted two studies that were part of our building and testing the theory of three worlds of mathematics to the concept of vector. The research was carried out at a Comprehensive School during the first part of the academic year September 2000–July 2001. The school has a good academic reputation (for example, in 2001, 63% attained a grade C or above in mathematics in the GCSE examination taken at age 16, as compared with a national average of 54%). The research involved 23 Lower Sixth students (aged 16-17), 26 Upper Sixth students, (aged 17-18), 2 teachers of Physics, and 4 teachers of Mathematics (two covering the preliminary work on vectors at GCSE level, and two teaching the two-year 'A' level course in the Sixth Form).

The research method draws upon qualitative and quantitative data and includes lesson observations, standard class tests to assess progress, and a specially designed conceptual questionnaire coordinated with clinical interviews with students and Mathematics and Science teachers (Ginsburg, 1981; Swanson *et al*., 1981). The data was triangulated, by analysing the books used by students and teachers, by videoing and observing classes, by interviewing teachers on their preferences and their expectations of the students' knowledge about vectors, with particular emphasis on the questions used in the conceptual questionnaire.

As a feature of the timing of the curriculum, the second year (Upper Sixth) students meet the topic of vectors earlier in the term than the first year. For this reason, the first experiment took place with the 26 Upper Sixth students. These were due to review the concept of vector in mechanics at the beginning of the year, having met it before from the text-book of Pledger (1996) two years earlier, and in mechanics in the previous year.

Despite this previous work on vectors, the students had significant difficulties with the concepts, some of which have already been mentioned earlier in this article. For instance, 25 out of 26 students were able to solve a problem involving horizontal and vertical forces (figure 2), but only 4 out of 26 were able to specify the forces acting on a block on an inclined plane and resolve parallel to the plane.

These 26 students were divided into two classes. One class of 11 students followed the traditional course whilst the other 15 were taught by the first-named author using full-class plenaries. The aim of the plenary sessions was to widen the students' perception of different contexts in which vectors were taught and to become more consciously aware of the core mathematical concept of vector and of its use in these various contexts. For example, it was sensed that force is essentially a position vector because it acts on a specific point, while the velocity vector of an object such as a car would apply to all points on the car, provided that the car was going in a single direction. In this way, the idea of 'equivalent vectors' was linked to physical concepts such as the velocity or acceleration of a body moving in a straight line.

In each lesson, the starter session set the scene, often with groups of students being given transparencies to write down the main points that arose in the core part of the lesson. In the final plenary session of the lesson, each group of students presented their ideas to the class as a whole, for discussion. At this stage, the teacher and the other students were free to ask any questions they considered relevant, including explaining not only the procedures they used, but also the essential nature of their thinking. The teacher's task was to guide the discussion so that the whole class could develop a good perspective of the topic and make flexible links between various aspects of the problem under consideration. Although the teacher had a specific agenda to follow, there was ample flexibility allowed to follow up particular aspects that arose naturally in the discussion. For instance, during a plenary discussion on vectors, it became apparent that some students were restricted in their conceptual development by their earlier experience in various contexts. Several students had difficulties with free vectors. One of the students thought that the vector acting on an object can be only resolved into two components, both also acting on that object at the same point. He sensed that if the vector was moved away from the object, it will no longer act on that object and if the components were resolved using the triangle law, that the position of the other sides determined how the components operated. He was one of those who represented the components of gravity acting on a block on a sloping plane as the sides of a triangle (figure 4), in such a way that he was convinced that the component on the side of the triangle parallel to the slope was too far away to have any effect on the force down the plane.

It was during this first experiment that Joshua explained his idea of 'effect'. We decided to build on this notion with the first year students in our second experimental investigation later in the term. The work with the first year students began with a focus on physical actions to represent translations. Students were invited to take part in activities involving moving their hands around on a large piece of paper placed on the blackboard. First a volunteer was invited to place his hand on the paper and draw round the outline (hand X in figure 11), then to shift his hand to another position (A) without rotating his hand and again trace round the outline. The students were asked to suggest ways in which the translation from X to A could be represented in different ways. They soon realised that different groups chose different parts of the hand to draw an arrow showing the movement. One group suggested tracing the path of the little finger (arrow 1), another suggested an arrow denoting the movement of the middle of the hand (arrow 2). None of the groups showed the arrow away from the hand. The teacher intervened and asked if anyone could suggest a method that did not mark the movement of a point on the hand. A third group suggested an arrow of the same magnitude and direction that was not attached to the hand at all (arrow 3).

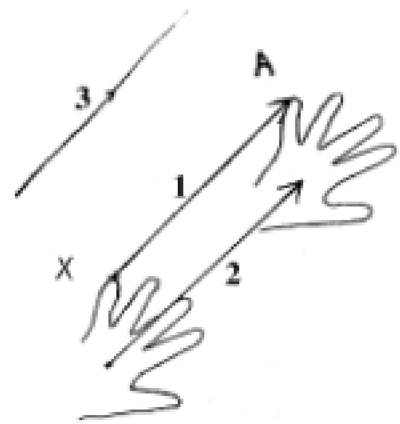


Figure 11: arrows representing the translation of a hand

Next, a student was invited to place his hand on the position of hand A and translate his hand to position B (figure 12). Invited to represent this shift by an arrow, he drew a directed line marking the movement of the thumb (arrow 4).

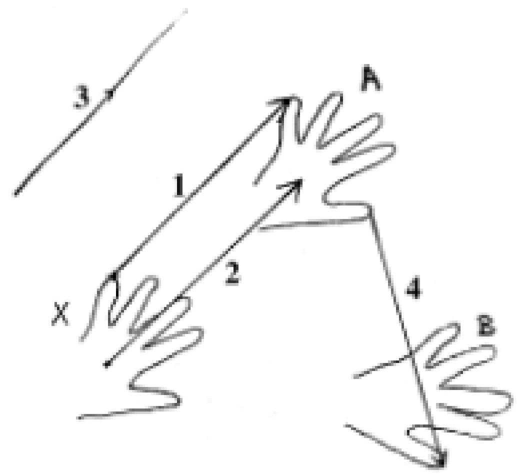


Figure 12: a further translation from A to B.

The teacher asked the students to draw a representation of the translation from position X to B, and this was drawn from the join of finger and thumb. (Figure 13, arrow 5.)

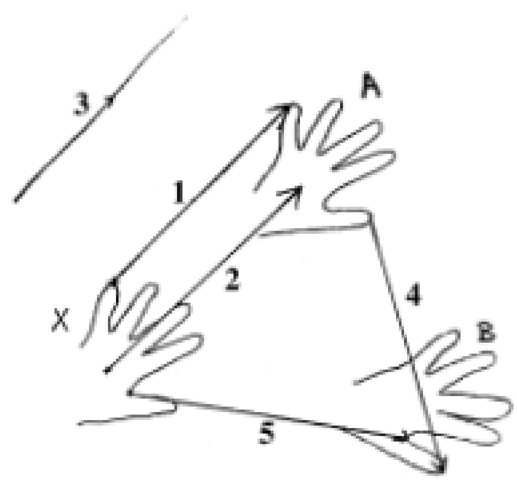


Figure 13: The combined effect (arrow 5)

The teacher now introduced the main idea. How could the translations be used to represent the translation from position X to position B? The students were invited to discuss various ways of doing this, and provided a solution by drawing lines XY, YC parallel to vectors 2, 4 respectively, which meet nose to tail at Y (figure 13). This use of equivalent vectors was further discussed to ensure that all students sensed the fundamental idea that any equivalent vectors could be used to perform the addition.

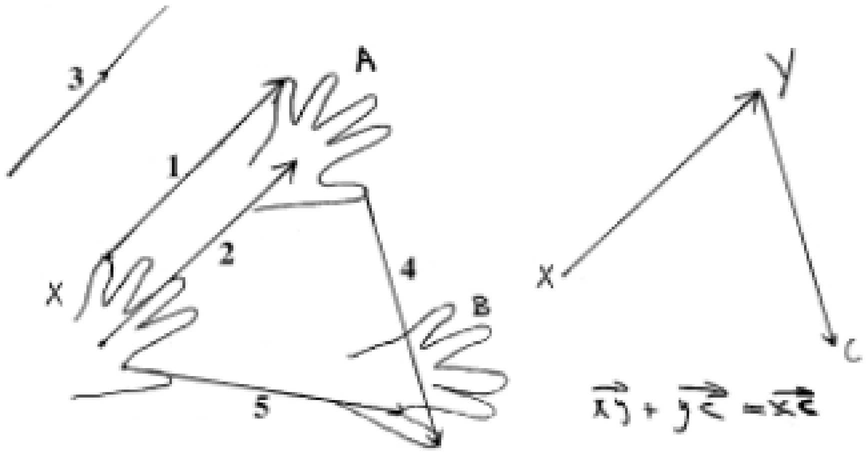


Figure 14: sum of (free) vectors

The students then sketched the horizontal and vertical components of the vectors in question and checked the computation using vector addition (figure 15).

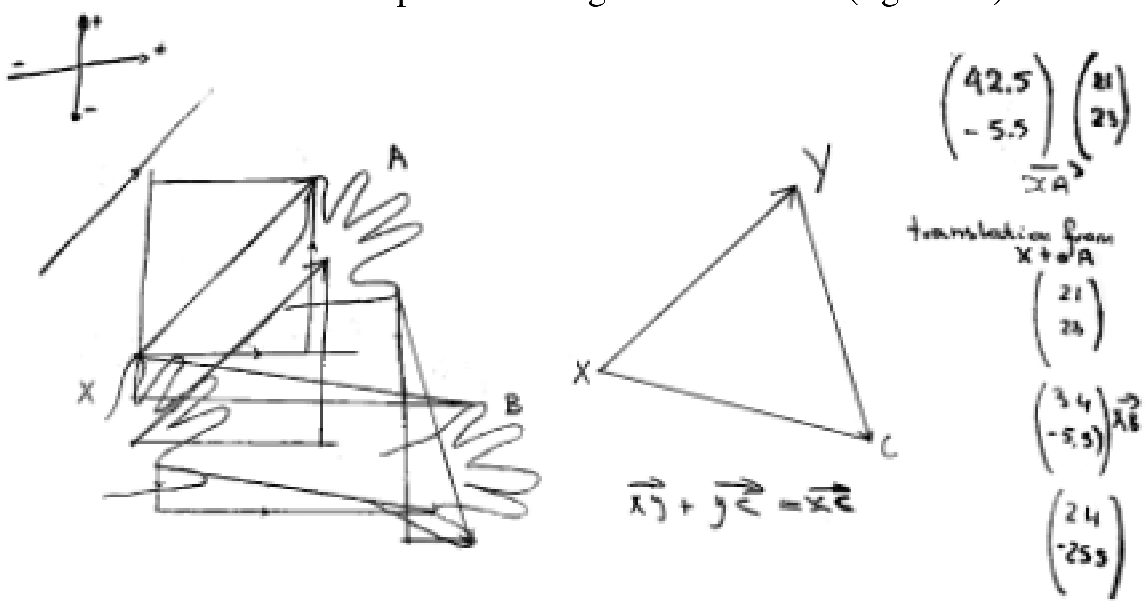
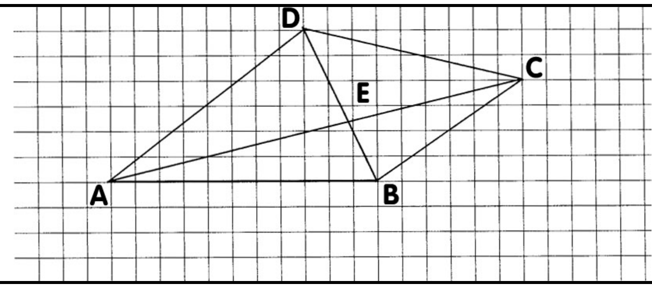


Figure 15: The final picture of the students' work.

Experimental Data

All the students participated in a test that included a question on their ability to add vectors represented on squared paper (figure 16). This allowed them to use the elementary method of counting squares, equivalent vectors, or any other methods they desired. The correct responses are given in table 1a, 1b. In brackets are the numbers of correct solutions obtained by counting squares.

In part (i), all the students wrote the correct answer straight down with no counting of squares. This shows the powerful effect of the triangle law as a journey following end-to-end with one arrow after another.



Show clearly a vector equivalent to:

(*i*) $\overrightarrow{AB} + \overrightarrow{BC}$

(*ii*) $\overrightarrow{AB} + \overrightarrow{AD}$

(*iii*) $\overrightarrow{AB} + \overrightarrow{CA}$

Fig. 16: Testing the visual sum of two vectors

Upper 6th	Reflective (N=15)	Standard (N=11)
Question (i)	15(0)	11(0)
(ii)	8(1)	1(1)
(iii)	10(3)	6(6)

Table 1a: effect of reflective plenaries

Lower 6th	Embodied (N=7)	Standard (N=13)
Question (i)	7(0)	13(0)
(ii)	5(3)	3(3)
(iii)	6(4)	5(3)

Table 1b: effect of embodied approach in reflective plenaries

Table 1: total correct (figures in brackets are those counting squares and adding cooordinates)

Part (ii) is more subtle. Only 1 out of 11 of the upper sixth standard students and 3 of the 16 lower sixth standard student gave a correct response, and *all* of these did it by counting squares. On the other hand, 8 out of 15 upper sixth reflective students and 5 out of 7 lower sixth following the embodied approach were successful and they included students who solved the problem using equivalent vectors (7 out of 8 in the upper sixth, 2 out of 5 in the lower sixth). Looking more closely at the errors committed reveals further interesting data. The picture may *look* like an example of the parallelogram rule, but the figure is not a parallelogram. The two lower sixth 'embodied' students who made an error both used the parallelogram erroneously to give the answer AC. In a sense, they used a correct conception with a mistaken interpretation. *None* of the other students made this error. Instead *all* of the students in the upper sixth with incorrect responses (7 out of 15 in the reflective group, 10 out of 11 in the standard group) gave the answer BD. This data suggests that the emphasis on the triangle law is having a negative effect that was not eradicated by the upper sixth plenaries. There was, however, a change in the lower sixth plenaries with their greater emphasis on equivalent vectors.

Part (iii) has another twist in the tale. The sum $\overrightarrow{AB} + \overrightarrow{CA}$ does not make sense in the naïve meaning of a journey. Travel from A to B then C to A requires a jump from B to C. To solve the problem therefore requires either a simple level solution by counting squares for the components and adding them, or the use of equivalent vectors. A third possibility is to use commutativity of addition to write $\overrightarrow{AB} + \overrightarrow{CA}$ as $\overrightarrow{CA} + \overrightarrow{AB}$ and use the triangle law to get $\overrightarrow{CB}$. This subtle idea did not occur explicitly in any written solution although it may (or may not) have figured in the thinking of 2 lower sixth standard and 2 lower sixth 'embodied' students who wrote the answer without any working. The standard students in the upper sixth who got it correct (10 out of 15) *all* explicitly counted squares, as did 3 of the 5 standard students out of 13 in the lower sixth. There was a greater variety of responses from those involved in plenaries who used both counting and equivalent vectors. The errors amongst the students mainly involved marking the vector as $\overrightarrow{BC}$, showing it as the third side of the triangle, but not taking the direction into account.

Considering only those students who were able to solve all three problems, we get the data in tables 2a and 2b. Considering the standard pupils first, we see that there is hardly any difference between first and second year sixth. Only one student in each case was able to get all three questions correct and this was done by counting squares. Amongst those involved in the reflective plenaries, in the upper sixth revision course, 7 out of 15 were correct on all three, showing some, but not fully satisfactory improvement. In the smaller group in the lower sixth, 5 out of 7 were correct on all three. The two making errors were those who had the right idea of using the parallelogram rule in part (ii), but misinterpreted the figure as being a parallelogram.

Upper 6th	Reflective	Standard
All 3 correct	7	1
Other	8	10

Table 2a: effect of reflective plenaries

Lower 6th	Embodied	Standard
All 3 correct	5	1
Other	2	12

Table 2b: effect of embodied approach in reflective plenaries

Interviews with six selected students in the lower sixth (three following the standard course, three following the embodied approach) confirmed that students following the standard course had problems adding two vectors that did not follow on one after the other. For instance, on being shown two vectors joined head to head, two thought that the resultant would be zero, because they would cancel out. The students following the embodied course did not have these problems because they were fluent in their ability to shift free vectors around to fit one on the end of another.

We believe that this evidence is consistent with two observations. First the introduction of reflective plenaries teasing out the meanings of free vectors in the second year revision course improved the flexibility of student conceptions of vectors. Second, the introduction of an embodied reflective approach in the first year gives the chance of building an even better level of flexibility in those use the approach from the beginning of the course. Follow up studies in the next year are now under way.

REFLECTION

The study so far has revealed the complexity of the meaning of vectors as forces and as displacements and the subtle meanings that are inferred in differing contexts. Studies in science education have attempted to build a classification of misconceptions without clearly identifying the underlying problems. Our approach is to develop a pragmatic method that will work in the classroom. One aspect is the use of conceptual plenaries, which are already becoming part of the formally defined curriculum in England. The other is to link physical embodiments to mathematical concepts via a strategy that focuses on the effects of actions. Our experience shows that such an approach can be beneficial in the short-term and we are continuing our practical and theoretical developments over the longer term.

Our experiment continues into a second cycle with a refined version of reflective plenaries building on embodied concepts. Curriculum constraints do not allow us to incorporate the full notion of Euclidean constructions in adding vectors geometrically, but we will introduce a new element in an embodied operation of shifting vectors maintaining the direction by moving a setsquare flush to a fixed ruler. This, of course, increases the possibility of students learning yet another procedure to be able to cope with the problems encountered. However, we are growing in confidence using reflective plenaries to encourage a rich global awareness of the relationships between a variety of embodied concepts and a symbolic concept dealing dually with vector as process and vector as concept. In this way we are building evidence that our approach linking embodiment to symbolism has great promise for the future, not only in handling vectors, but also in the wide range of other topics in which a variety of real-world situations are translated into symbolic mathematics.

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