Chapter 1

Thesis Overview

'I hear, and I forget; I see, and I remember; I do, and I understand.'
saying of Confucius, quoted in Nuffield (1967a)

'I do and I understand ***how****, I reflect and I understand* ***why****.'*
extended by Anna Poynter (2003)

Prologue

As a teacher of mathematics for many years I became concerned how students seemed to be able to learn to do techniques to score highly on examinations, yet seemed not to be able to apply their knowledge in slightly different situations, or to retain their skills for ready use in subsequent courses. For example, in mechanics my students were highly successful at resolving vectors horizontally and vertically to solve mechanics problems, yet when faced with a rectangular block on a rough inclined plane, most were unable to resolve the force due to gravity down the plane.

Up to this time, I had taught vectors according to the text-books, including physical experiments with objects on a slope, practising all possible calculations and variations. I made sure to warn students about any pitfalls and drilled into them the techniques of answering questions. They seemed to be able to answer every question in the book, until they met something slightly different on the test.

After talking with some of the students and teachers, who seem to have the same problem, I decided to take the matter seriously. I thought to myself: "If only students could concentrate on the simplicity of the mathematical idea instead of the many complications connected to different contexts, then they should be in a better position to solve problems in novel situations." However, although the mathematics seems to be simple for an expert, there seem to be untold pitfalls for the learner.

My quest in this thesis is therefore to understand better ***why*** *students have such difficulties and* ***how*** *they may be encouraged to reflect on their knowledge to focus on the essential ideas in a way that could be used flexibly in new contexts.*

1.1 Introduction

This study is concerned with how students make sense of mathematical ideas and how they symbolise their physical conceptions in meaningful ways. Although it focuses on the notion of vectors, the theory constructed is intended to have a much wider applicability, linking cognitive theory with practical application in the classroom.

Mathematics is usually considered as a highly logical activity but it is created and learnt by human beings who interact with the world and develop higher-order thinking capacities by reflecting on their actions. In practical terms this suggests two distinct stages: encouraging students in ‘hands on’ specific activities, and then to reflect on the essential elements of these activities to build theories. In theoretical terms the work relates to the cognitive science literature of embodiment and the mathematics education literature which focuses on how processes that we perform can be symbolized so that the symbols can become meaningful ideas that we can manipulate and think about.

The notion of embodiment has rich and varied meanings in the literature of cognitive science, which will be discussed in greater detail in chapter 2. The intention of this thesis is to develop a practical theory to enhance learning by reflecting on physical experience to build sophisticated mathematical concepts.

The concept of the vector is an important and useful notion in science and mathematics. We encounter it as both a geometrical and a symbolic idea. The notion of vector in the English curriculum initially appears mainly in the science syllabus. The first contact with vectors that English students have is in Physics, based on real world experience and observation. Physical experiences, however, occur dynamically in time, while the problems set in Physics and Mathematics are often considered at a specific instant of time. Everyday experience may tell a person what happens eventually but not what happens at an instant, resulting in what I shall call a ‘false intuition’ that can potentially impair an individual’s capacity to think logically and mathematically.

Real experiences with vectors are many and varied (displacement, velocity, acceleration, forces, etc.) and the differences can cause complications, whereas the mathematics of vectors, once it is understood, has a simpler structure that applies to *all* embodiments and therefore potentially has more power. The view I have held for many years, in the case of vectors, is that there are two fundamentally different worlds: the *embodied* world with its various different environments (contexts) that involve sensory experience and visualisations, and the *symbolic* world of mathematics with its use of symbols to represent vectors with their components being written in a column matrix or as **i** and **j** components. There are parallels between these worlds in the way they compress action into process and into concept which we shall discuss in greater detail in chapters 2 and 3. However, in the school environment, the emphasis is on preparing students for assessment and there is always a dilemma of how to approach the teaching of the topic in a crowded curriculum. There often does not seem to be time to show students how these two parallel words work together and enable them to construct the concepts in a meaningful way. It is easy in a crowded syllabus to make wrong assumptions about students' embodied awareness of the topic and enter some of them into the symbolic world too quickly.

In the development of the research in the thesis, a preliminary study helped to build the theoretical hypothesis. I decided to involve the group of students I was teaching, in activities and reflective plenaries in which I focused on how the mathematical notion of vector might be built from the physical experience.

Our specific practical activities involved conceiving transformations as physical movements of a shape on a flat surface and focusing on the idea that different actions can give the same result. For instance, when looking at the hand translating a shape, the end of each finger moves in the same direction by the same amount, thus the arrows from the starting point to the finishing point all have the same magnitude and direction and effectively represent the same translation. By focusing on the *effect* of the translation rather than the physical movement of a particular point gives the student an opportunity to give embodied meaning to the mathematical notion of a *free* vector (with specific magnitude and direction but not starting at any specific point in space).

This approach has the potential of being extended to the mathematical notion of addition of vectors in that the *effect* of two successive shifts is the same as that of the single shift from the initial position to the final position. This in turns leads the concept of the commutativity of vector addition, which only makes sense when we think of *free* vectors detached from any context. (If *A*, *B*, *C* are fixed points, then the journey $\overrightarrow{AB}+\overrightarrow{BC}$ makes sense as a journey from *A* to *B* to *C*, but $\overrightarrow{BC}+\overrightarrow{AB}$ suggests first travelling from *B* to *C*, and then seems to require a jump from *C* to *A* to perform the second journey from *A* to *B*.) By using a combination of practical activity, reflection and plenary discussion, it becomes possible to create an environment in which students can potentially construct links between different ways of looking at the same concept in different contexts. The time devoted to the topic of vectors was the same as in other classes, but the intention was to produce more long-term stability of concepts.

Concepts of vectors are usually formulated in specific contexts, such as displacement and forces in the physical world, where vectors either follow each other (displacement) or come out from one point (forces acting on an object). Free vectors in the mathematical world, on the other hand, are often drawn as separate vectors in space which do not overlap each other. Each context therefore has its own type of general format which will be called a *generic case* in that context. In the context of a journey, a combination of journeys is given by following the first journey by the second, and in the generic case, the first journey ends where the second begins. In the context of several forces acting on a point, a combination of forces is added together symbolically by working out the horizontal and vertical components of each vector and adding them together to present the final answer as a column vector. Graphically, students would be expected to add free vectors ‘nose to tail’ by shifting the start of one vector to the end of the other. However, if they are given example of vectors that cross, or vectors whose ‘noses’ meet at a point, such questions might cause confusion.

These examples are called *singular cases*. In simple terms, a singular case is an example that has incidental properties which are not typical of the general case in the given context. It is my belief that students' ability to handle singular cases will be highly indicative as to whether they have a rich flexible concept rather than a more limited procedural view.

Examinations usually involve generic types of problems and, through extensive practice with past questions, students can be very well prepared to answer such problems. They learn the procedures for the specific situation; however, even students who obtain a top mark in this way in the exam might not be able to answer unfamiliar singular cases. From my experience, when seemingly very capable students come out from an exam and say that a particular question was impossible, by looking at the question afterwards I was often able to classify it as a singular case.

To encourage students' construction of rich connections between different physical and mathematical contexts, I built reflective plenaries into lessons. The intention was to help each student to build a concept of vector as a cognitive unit which encapsulates all the aspects of free vector into a simple single mathematical idea. The goal was to build a long-term conceptual stability of concepts to give students the insight and confidence to tackle singular cases and solve them using understanding at a higher cognitive level.

1.2 The background to the research

I initiated my investigation by looking carefully at the Mathematics and Physics syllabuses and talking to the teachers of both subjects. This seemed appropriate since both subjects use vectors in various topics.

Through discussions with the teachers I found that other teachers shared similar strategies to those that I used, with a positive attitude in trying their best to help students to perform well in exams. Like myself they show awareness of students' difficulties and, in their teaching, do their best to give students the right procedures to answer questions and ways of overcoming the obstacles. They have no time to find the reasons for problems students might encounter but know instinctively which questions they should find easy and which they may find difficult, and what type of mistakes they might make.

My goal was to attempt to find the nature of the students' difficulties, to formulate general ways of teaching that would improve the student's fundamental insight into mathematics and to express this in a language that could be shared both with other teachers and with the students themselves.

In attempting to make headway in such an ambitious task, I sensed that it would be necessary to take account of positive aspects (marked below with a 'smiley' ☺) and negative aspects (marked with a frown ☹)

These aspects were as follows:

☺ In their quest for improving the students' understanding and to give them additional meaning to the ideas they wish the students to learn, teachers often try to give the students a physical experience of the idea by either involving them in doing their own experiments or by teacher demonstration of a physical experiment.

☹ In the case of vectors, experience reveals that this can cause additional problems instead of solving them. When talking to the Physics teachers, it appears that different contexts are treated differently in that subject. For example, vectors as displacements are considered as journeys following each other and, when adding two displacements, the triangular rule of addition would be considered (figure 1.1). On the other hand, vectors as forces are usually presented as acting on an object (presented as particle), and for addition of two forces the parallelogram rule would be considered (figure 1.2).

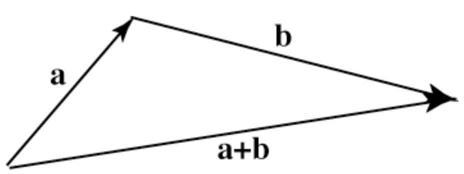

Fig. 1.1 The Triangle Law

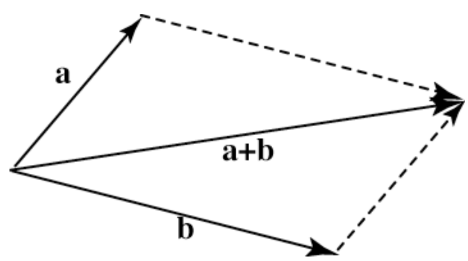

Fig. 1.2. The Parallelogram Law

Alternatively students' would be taught to resolve the vector quantities into horizontal and vertical components, which would be added together. The mathematics teachers involved in this research have the intuitive notion that some questions might be difficult because of the physical representation (singular cases) but generally for them the vector is a cognitive unit which can be applied to different contexts. They do not seem to consider that different contexts have clashing meanings.

☺ In attempting to help the students to build a more flexible conception of the notion of vector that includes the whole structure of embodiment and process-object encapsulation, I was struck by the interpretation formulated by one particular student whom we will call Joshua. He explained that different actions can have the same '*effect*'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors. He also observed that solving problems with velocities or accelerations is mathematically the same: "the only difference is that one is metres per second and another metres per second squared." He was able to operate with the vector as a cognitive unit which can be applied to different contexts.

☺ This idea of 'effect' seemed a possible way of introducing vectors in a mathematical way that focused on the mathematical ideas rather than the different physical contexts that lead to 'false intuitions'. The transformations of objects can be seen as actions on physical objects, but then, by focussing on the *effect* of the transformation, it may be possible to give a meaning of the transformation of the object as a mental concept. Such a concept is already available: it is the physical arrow that represents the magnitude and direction of a translation. In this way, the transformation *as an action* can be related to a vector *as a mental object*. To focus positively on these ideas, I decided to involve the students in physical activities which were then used as the basis for reflection and discussion in reflective plenary sessions, with the teacher helping students to build theoretical ideas based on their own experience.

1.3 The structure of the thesis

The relevant literature is reviewed in chapter 2. As a major focus of the thesis is the transition from physical experience to mathematical symbolism, it is important to look at the literature as to how intuition can affect understanding and how the experience from the physical world can be used to support conceptual development instead of having a negative effect. The chapter also surveys different theories of knowledge and learning, focusing not only the ways in which knowledge is constructed and understood, but also how it is compressed and encapsulated as thinkable symbols in the transition from embodiment to symbolism.

Chapter 3 begins with a study of the curriculum in Mathematics and Physics that is encountered by the students in their earlier studies, focusing on the development of the concept of vector in the text-books that have been used. In the same chapter, I also look at some examples of research conducted in Mechanics and into vector concepts that are relevant to this research.

Chapter 4 reports results from a preliminary investigation which took place before the main research. This enabled me to formulate the central theoretical hypothesis to be tested in the body of the thesis and to design a theory of teaching vectors to use and study in the experimental work with students. The fundamental framework is illustrated in figure 1.3, beginning from (physical) embodiment with actions on objects, to focus on the effects of those actions to form mental concepts.

action (doing)	➔	effect (seeing, reflecting and focusing)	➔	mental concept (symbolising)

Fig. 1.3 compression from action to concept by focusing on the effect

The details of this preliminary investigation and how it led me to the idea of *effect* are discussed in detail in chapter 4. Having focused on the important elements to be considered in the research study, chapter 5 turns to a consideration of the methodology of the research and the specific methods to be used. These are trialled in a Pilot Study described in chapter 6.

The Pilot study focused on carrying out the teaching experiment and designing and testing a questionnaire to assess the comparative effects of the experimental method used in an experimental class, compared with a traditional approach in a parallel control class.

The questionnaire consisted of questions designed to test:

- if students compressed the notion of free vector and the *effect* of actions to see the equivalence of vectors having the same magnitude and direction which could be freely represented anywhere in the plane whether touching the object being shifted or not;
- if students develop a mental concept of vector capable of solving not just generic but also singular cases;
- if students have the flexibility to use a vector as a mathematical mental concept to solve problems independent of the context;
- if they can apply the concept of the commutative law of addition: **u**+**v**=**v**+**u**, which can only be understood by students treating **u** and **v** as 'free vectors'.

The pilot study also included trial interviews to test some of the interview techniques intended to seek greater insight into:

- use and flexibility of language when discussing problems connected with vector addition;
- the focus of attention at any given time (whether it is on actions, or procedures or on the effects of those actions and procedures);
- the way in which different contexts affect their thinking;
- their flexibility in dealing with different modes of operation (graphical/symbolic).

The details of questions to be used in the main study, which were tested in the Pilot study, and the outcome of the Pilot Study are presented in chapter 6.

Chapter 7 presents and analyses the quantitative data collected from the use of the questionnaire in three tests: pre-test, post-test and delayed post test. Chapter 8 presents the qualitative data from interviews with the teachers. Chapter 9 presents the qualitative data from the interviews with selected students. The data collected gives both quantitative and qualitative evidence that is consistent with the main hypothesis that the rich embodied experience in the experimental approach, focusing the students on the notion of effect, helps them to internalise processes into manipulable mental concepts that remain stable through to the delayed post-test.

The final conclusions, including detailed summaries of the research and analysis of results, are presented in chapter 10, together with reflections on the limitations and generalizability of the results, leading to avenues for future research and development.