Chapter 2
Literature Review

2.1 Introduction

This chapter presents the review of the literature that theoretically underpins the thesis. It includes general theories of learning and understanding, but focuses particularly on embodiment on the one hand and symbolism on the other. The relationship between embodiment and symbolism will play a fundamental role in the approach to the teaching of mathematics in general and vectors in particular. It will be used to build a theoretical framework in which meaning for symbolism is constructed from reflection on embodied activities, and to lay out a schema of development to enable the cognitive development to be described and tested in the main study.

2.2 Theories of knowledge and understanding

In working with students, I found that their responses did not seem to fit within a single theoretical position and therefore found it necessary to review a number of different theories to build a theoretical framework to categorise answers that arose in my research. The framework developed is a blend and extension of other theories. In what follows I describe the literature and the theories I have considered, and my reasons for building the theoretical framework used in this thesis.

2.2.1 Intuition

From my experience, different physical encounters of vectors gained in Physics or every-day life can cause complications for students. They answer questions from a 'false physical intuition' point of view. For example, when I asked students in the preliminary study to add two vectors **a** and **b** as shown in fig 2.1, nearly 50% gave a wrong answer **c,** marked with a dotted line.

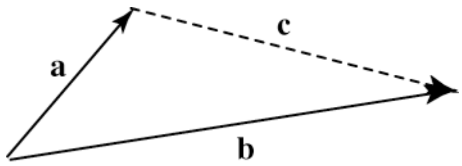

Fig. 2.1 Example of the 'intuitive' addition of two vectors

Although this might seem like a misapplication of the triangle law, from interviews, it appears that several students used a physical experience of two people pulling them in directions of vectors **a** and **b**. There is a stronger pull in the **b** direction and therefore that's where they are going to end up moving. Students in this case seem to forget about the mathematical rules of adding vectors and base their answer on 'physical intuition' which, regrettably, leads them astray. As I have decided to classify such answers as physically intuitive responses I have become interested at the ways that 'intuition' has been formulated in the past.

The early philosophers were interested in intuition as a basic human faculty. In his '*Essay Concerning Human Understanding'* (1690), the English Philosopher, John Locke specifies three degrees of knowledge, which are "intuitive", "demonstrative" and "sensory". In discussing Locke's ideas, Sierpinska (1990) refers to intuitive knowledge as "irresistible and certain". Intuitive knowledge seeks "identity or diversity" because "it is the first act of the mind to perceive its ideas and to perceive their difference and that one is not the other" (Sierpinska, 1990, pp. 28–29).

In his *Critique of Pure Reason* (1781), the philosopher Kant summarizes cognition in the following terms:

> […] all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas. (Kant, 1751, p.404)

Skemp (1971) specifies two modes of functioning of intelligence: intuitive and reflective. He specifies the intuitive mode as being 'aware through our receptors (particularly vision and hearing) of data from the external environment, this data being automatically classified and related to other data by the conceptual structures,' (Skemp, 1971, p.51).

Royce *et al*. (1978), in a review of psychological epistemology, includes intuition as a "distinct cognitive phenomenon, together with perception, thinking and symbolisation."

Fischbein, Tirosh and Melamed (1981) write:

> Accepting intuitively a certain solution or a certain interpretation means to accept it directly without (or prior to) resorting explicitly to a detailed justification. [...] The problem of identifying the natural intuitive biases of the learner is important because they affect – sometimes in a very strong and stable manner – his concepts, his interpretations, his capacity to understand, to solve and memorize in certain area. We are naturally inclined to retain interpretations which suit these natural, intuitive biases, and to forget or to distort those which do not fit them.
>
> (Fischbein *et al*., 1981, p 491)

They end their article by concluding that:

> Didactical strategies must be devised for shaping improved intuitive interpretations and for overcoming conflicting intuitive biases
>
> (Fischbein *et al*., 1981, p. 512)

Fischbein (1994) specifies intuition as one of the three components of mathematics as a human activity. The other two components are formal and algorithmic. Theoretically, intuitions may play a facilitating role in the instructional process, but very often, contradictions may appear:

> Intuitions may become obstacles – epistemological obstacles [...] – in the learning, solving, or invention processes.
>
> (Fischbein, 1994, p. 232–234)

Sierpinska (1990) summarises a model of understanding mathematical concepts developed by Herscovics and Bergeron (1989) in which they also look at intuition.

She quotes them:

> Intuition [...] arises from a type of thinking based essentially on visual perception and results in an ability to make rough non-numerical approximations. (Sierpinska, 1990, p. 28)

According to Dewey (1988), and then Piaget, the first stage of concepts are formed from experience of a single object and by building a general category of objects with similar or the same characteristics. The second stage comes from discriminating between properties of characteristic and non-characteristic objects. The third stage consists of “application to explaining new cases with the help of a discovery made in one case,” (op. cit., p. 164-165). In my own research, being aware of the possible ‘false intuitions’ in the second stage, the question arises whether the third stage—if implemented carefully and reflectively—can help to straighten the misconception gained in the second stage.

2.2.2 Instrumental-relational understanding

As I was intending to reintroduce the concept of vector addition to the experimental group of students in a specific context, I decided to look at the theory of the instrumental and relational understanding of Skemp (1976), which was expanded by other researchers, and also at the related theory of procedural-conceptual knowledge introduced by Hiebert & Lefevre (1986).

Skemp (1971, p. 15) describes two types of learning. One he calls habit learning, or rote-memorizing, which is instrumental. The other learning involves understanding, which he calls intelligent learning. Piaget pioneered studying the second type of learning (cognitive processes in children and adults). Skemp indicates that reflective activity “involves becoming aware of one’s own concepts and schemas, perceiving their relationships and structures, then manipulating these in various ways,” (Skemp, 1971, p 77). He also suggests that “low-order concepts can be formed, and used, without the use of language,” (p. 26) however “making an idea conscious seems to be closely connected with associating it with a symbol,” (p. 78) and “it is largely by the use of symbols that we achieve voluntary control over our thoughts,” (p.78). According to Skemp, symbols help us to “reduce the cognitive strain of keeping the whole of the relevant information accessible,” which is very important since “one of the aims of reflection is to become aware of how one’s ideas are related, and to integrate them further.

My analysis of observing students for many years, as a mathematics teacher, is that the lack of the requirement for analysis and symbolic accuracy in graphical representations causes many problems when students have to apply their knowledge to questions involving applications of vector quantities in two and three dimensions. The lack of accuracy seems to often stop, observed by me students, relating ideas and integrating them further. Krutetskii (1976) suggested that gifted children have "a tendency to interpret environmental phenomena on the level of logical and mathematical categories, to perceive many phenomena through the prism of logical and mathematical relationships, and distinguish a mathematical aspect when perceiving many phenomena in an activity" (1976, p. 302).

Van Hiele (2002) writes, 'The theoretical level to which the axioms belong can only be reached by starting from the descriptive level' (object recognition level) otherwise they have to learn 'parts of geometry by heart and that means only instrumental understanding'. He also states that 'Many teachers were very content with such a course of events [...] and there were always pupils who liked mathematics from the very beginning and found their own way to the higher levels. But a great part of the pupils developed a dislike of geometry and after their study was finished forgot practically all of it.' (van Hiele, 2002, p 34-35). He also warns that, 'Reflection fails because the pupil only disposes of concepts of the visual level and those concepts do not lead to a result on the descriptive level,' (van Hiele, 2002, p. 35). The visual level means that shapes are recognised by seeing and not by their properties. He gives as the instrumental example drawing a picture using coordinates or vectors on the squared paper.

2.2.3 Procedural-conceptual knowledge

According to Hiebert and Lefevre (1986), procedural knowledge "is made of two distinct parts. One part is composed of the formal language, or symbolic representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical task." (p 6). The second one of these seems like

Skemp's instrumental understanding, which indicate step-by-step instructions that prescribe how to complete tasks. Hiebert and Lefevre say, "in general, knowledge of the symbols and syntax of mathematics implies only an awareness of surface features, not a knowledge of meaning," (p. 6). What the authors underline is that conceptual and procedural knowledge have to be linked, otherwise "students may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing," (p. 9). Expanding on their ideas of symbols, they say, that symbols represent ideas that can be met in real-world experiences. These ideas can be represented as conceptual knowledge, which provides the referents for symbols.

This would fit with the way I reintroduced the experimental group to the idea of the vector (described in methodology chapter, later on).

> If the procedures are related to the underlying rationale on which they are based, the procedures begin to look reasonable. It is possible to understand how and why the procedures work. [...] procedures that are meaningful, that are understood by their users, are more likely to be recalled. (Hiebert and Lefevre, 1986, p. 10-11)

Therefore, if my technique of reintroducing the experimental group to vectors is correct, the retention should be better and students should be able to perform better than the control group after a break (six months in case of my research).

Basically conceptual knowledge and relational understanding indicate that somebody learnt something with meaning, while procedural knowledge and instrumental understanding indicate that somebody learnt how to solve a problem but not necessarily with meaning.

While the first ideas of Skemp's instrumental and relational understanding placed these two types of understanding into separate classifications, Hiebert and Lefevre (1986) say that

> Not all knowledge can be usefully described as either conceptual or procedural. Some knowledge seems to be a little of both, and some knowledge seems to be neither. (Hiebert and Lefevre, 1986, p. 3)

They write that

> [...] conceptual knowledge is characterized most clearly as knowledge that is rich in relationships (Hiebert and Lefevre, 1986, p. 3)

They also say that

> In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information.
> (Hiebert and Lefevre, 1986, p. 4)

They quote Skemp (1971) when describing "understanding" as "the state of knowledge when new mathematical information is connected to existing knowledge." (p. 4). The other way they see development of conceptual knowledge is by "the construction of relationships between pieces of information that already have been stored in memory". They quote Bruner (1961), Ginsburg (1977) and Lawler (1981) as the predecessors of such a theory. They use the term 'abstract' as the degree to which a unit of knowledge is tied to a specific context. According to them: "Abstractness increases as knowledge becomes freed from specific contexts," (p. 5).

This is very relevant to my investigation of students' responses. I have found from the preliminary investigation that students who performed best in the questionnaire, used in the main study, were those who either connected to a very specific context of a journey or those who saw the vector as a mathematical object. And therefore it seems that they used abstractness to different degrees.

Hiebert and Lefevre write (1986, p. 5) that "some relationships are constructed at a higher, more abstract level than the pieces of information they connect", which they call a reflective level. They note that it is not always easy to assess where procedural knowledge ends and conceptual starts. I have tried to assess this difference not only through the responses to my questionnaires but also through the interviews. The assessment of the students' responses is graded in stages of their conceptual development. These stages were developed with the help of the text-book that students study in Year 11. The book introduces vectors in stages from the embodied action of transformation of an object to the idea of the column vector, and arrows having a specific direction and magnitude, through to the idea of vector addition and equivalent vectors to the idea of the free vector. The interviews looked at the development of students' language to express their actions when adding vectors, trying to assess if they express their conceptual or procedural knowledge.

To obtain a deeper insight into the nature of human understanding, it has proved useful to look more closely at the link between intuitions produced by embodiment and the symbolism that is used to represent the processes and concepts.

2.3 Different modes of operation in mathematics

At school students are introduced to vectors in two ways. In Physics, vectors are introduced as journeys or forces, added according to the triangle law or parallelogram law, with different meanings and then represented as two one-dimensional components which are added by adding components. The sixteen-year-old students studying Mechanics in my school, who are taught in this way seemed to cope well with resolving horizontally and vertically and solving problems formulated in this context. However, according to my preliminary study (to be discussed in detail in chapter 4), many of these students have subtle problems with geometric interpretations, particularly with free vectors. The evidence arose in the way they answered certain 'singular' (unusual) questions (shown in figure 1.4b) which do not conform to the general prototypes that are suggested by their earlier experiences. In chapter 1, I theorised that if students are given embodied experiences focusing on the *effect* of transformations rather than the specific actions involved, then they have the potential to construct an embodied conceptualisation of the notion of free vector, and then cope more easily, not only with generic cases, but also with singular cases.

This requires a consideration of the literature that relates how physical experience with the outside world (embodiment) plays its part in the learning process. A major source for these ideas is the work of Lakoff and his colleagues who consider how human embodiment underpins abstract thinking.

I will also be looking at the importance of symbolic representations in the ability to model problems abstracted from the outside world in mathematical terms, and how reflection on mental and physical actions affects the building of coherent conceptual schemas. This involves considering not only how operations are carried out by sequences of step-by-step actions, but also how the effect of these actions can be symbolized and considered as mental entities to think about. A major source for these ideas is the theory of encapsulation of processes as mental objects as formulated by Dubinsky (1991) and Sfard (1991), and in the theory of *procepts* (where a symbol dually evokes either *pro*cess or con*cept*) formulated by Gray and Tall (1994).

2.3.1 Successive stages of cognitive development

Piaget (1985) describes cognitive development of the child in several stages: sensori-motor, pre-conceptual, concrete-operational and formal-operational. To underpin this development, he formulated a three-part theory of abstraction. In the first two stages (sensori-motor and pre-conceptual) a child goes through the process of *empirical* abstraction, when (s)he focuses on physical objects and their properties, noting similarities and differences that are abstracted empirically. In the third, concrete-operational stage, the child focuses on *actions* on objects and the properties of these actions result in what he calls *pseudo-empirical* abstraction. The formal-operational stage is described in his theory in terms of *reflective* abstraction in which 'actions and operations become thematized objects of thought or assimilation' (Piaget, 1985, page 49). He suggests that these stages of development apply to children from birth to about age of 12.

Piaget's ideas of conceptual growth were adapted by many researchers who developed them to apply to any age to formulate how conceptual growth takes place. Bruner (1966), for example, introduced three modes of representation: *enactive*, *iconic* and *symbolic*. He wrote:

> What does it mean to translate experience into a model of the world? Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. [...] There is a second system of representation that depends upon visual or sensory organisation and upon the use of summarizing images. [...] we have come to talk about the first form of representation as enactive, the

> second is iconic. [...] Finally, there is a representation in words or language. Its hallmark is that it is symbolic in nature.
> (Bruner, 1966, p. 10–11)

Bruner's enactive mode of operation is based on action and begins in Piaget's sensori-motor stage, to be followed by the ikonic and symbolic modes in the pre-conceptual and concrete-operational stages. In older individuals, all three modes may be available and used as appropriate in different contexts.

Lakoff and Johnson (1980) formulate their idea of conceptual embodiment as follows:

> Our experiences with physical objects (especially our own bodies) provide the basis for an extraordinarily wide variety of ontological metaphors, that is, ways of viewing events, activities, emotions, ideas, etc., as entities and substances. (Lakoff and Johnson, 1980, p. 25)

Lakoff & Núñez (2000) propose that all human ideas are grounded in sensori-motor experience. This involves the use of formulatable cognitive mechanisms by which the abstract is comprehended in terms of the concrete by using a *conceptual metaphor*. They claim that most mathematical thought makes use of conceptual metaphors. (2000, page 5). According to Lakoff & Núñez, human reason crucially depends both on human experience and imagination and therefore categorisation depends partly on human perception and motor activity, and partly on mental imagery.

2.3.2 Construction of meaning

Constructivists see students as active learners, who make sense of the world on the basis of the links between past experience and new information. In doing so, students may need to reconstruct their earlier conceptions to make sense of new information (Driver and Oldham, 1986). This process can occur only when students are dissatisfied with their current conception and feel the need for a new one. According to Posner et al. (1984) they should also consider the new concept as intelligible, plausible, and useful in solving problems.

However, a stumbling block for such a development and reconstruction can be what Lakoff describes as a ‘prototype effect’. He (1987) quotes from studies of Rosch that questions the belief that “categories are defined only by properties that all members share,” for if that were true, then

> [...] categories should be independent of the peculiarities of any being doing the categorizing; that is, should not involve such matters as human neurophysiology, human body movement, and specific human capacities to perceive, to form mental images, to learn and remember, to organize the things learned, and to communicate efficiently.
> (Rosch, quoted in Lakoff, 1987, p 7)

On the contrary, the research conducted by Rosch and others demonstrated that

> [...] categories, in general, have best examples (called “prototypes”) and that all of the specifically human capacities just mentioned do play a role in categorization. (ibid. p.7)

Early stages of mathematics in English primary school, are taught through physical activities using the senses and it is hoped that children will build on this experience to comprehend the nature of mathematical ideas, integrating them with their previous knowledge, and building a new category or concept or, where necessary, rebuilding the previous one. However, there is always the danger that pupils will accept a prototype (an example as representation of the whole category) as the concept. It is therefore important how we introduce our students to a new conceptual idea and to be aware of which context we are going to use for our explanations and discussion. According to Jaworski (1994)

> The pupil might fit the teacher’s words into her own experience to get a meaning different from what the teacher tried to convey. Because people interpret words and gestures differently, any attempt to convey knowledge in an absolute sense must be seen as quite likely to fail. A teacher therefore has to find ways of knowing what sense pupils make of the mathematical tasks which they are set, in order to evaluate activities and plan further lessons.
> (Jaworski, 1994, p. 220)

She describes the situation in her article in which

> [...] the activities in which the learners participated and encouraged them to be mathematical, that is to act as mathematicians by mathematising particular situations created by their teacher [...] learners shared perceptions with each other and with the teacher, and their ideas became modified or reinforced as common meanings developed. This enabled learners to become clearer and more confident about what they knew and understood. (Jaworski, 1994, p. 229)

In the case of vectors, the pupils' first introduction in typical English schools occurs in Science lessons, mainly through thinking about forces, which is a highly particular context with implicit properties that act as possible obstacles to the general notion of vector. Many pupils have thinking that is flexible enough to cope with the transition to the general notion of vector, despite the specific peculiarities of this particular embodiment. However, from the initial investigations into the topic of vectors, which will be described in greater detail in chapter 4, it seems that there are many more students for whom the concept of force becomes a prototypical concept of a vector and these pupils have a problem when the construction of the general concept of vector becomes necessary.

2.3.3 An example: the case of fractions

The subtleties required in construction of mathematical concepts can be illustrated by the case of fractions. Many mathematics text-books introduce the idea of the fractions as part of circles (pizzas, pies, cakes). This type of representation is very restrictive, and is only good for the imagination of simple fractions like 1/2, 1/4, 2/3, 5/6, etc. Kerslake's research (1986) shows that

> [...] the only model of a fraction that was widely accepted was that of a geometric 'part of a whole' Not only was it the only universally accepted model of 3/4, but children referred to parts of circles or parts of cakes when trying to explain other problems during the course of the interviews, such as addition of fractions, or whether 2/3 is bigger or smaller than 3/4. (Kerslake, 1986, p. 71)

Because of this representation

> [...] most children found it difficult to think of fractions as numbers, particularly when asked to place them on the number line.
>
> (Kerslake, 1986, p. 71)

After teaching an experimental group with a number line only, she concluded that

> [...] while the geometric 'part of a whole' model may well be useful one in establishing some of the basic ideas about fractions, serious consideration is necessary as to its limitations and to the need for presenting the idea of a fraction in a wider context.
>
> (Kerslake, 1986, p. 96)

A particular conclusion drawn from this research was that "the distinction needs to be drawn between the embodiment and the idea," (p 96).

This experience with fractions shows that a single embodiment of a general concept can inhibit the formation of a more general version of the same concept that has a wider range of application. The same problem seems to be happening in the case of vector. Experiencing a vector in a particular embodiment may lead to the student being able to operate in a limited range of cases that are similar to the students' experiences, but which are too narrow to cope with even slightly unfamiliar examples. These limitations may be revealed by presenting the student with 'singular cases', for instance the case where the resultant is required for two arrows whose heads are at the same point.

2.3.4 Embodiment of mathematical concepts in the physical world

Skemp used the word *embodiment* before it became fashionable in more recent theories of embodied cognition, to describe the way in which a mathematical concept is given a physical meaning that represents the underlying mathematical ideas in a clear and explicit manner. Skemp (1971, p. 176-177) gives an example of embodiment of equivalent fractions arising through the double operation of combining and sharing, which, in mathematical terms, are commutative. Sharing followed by combining gives the same result as combining followed by sharing, in the following sense. He gives example of a fraction 3/8 which we can first share the standard object (a rectangle) into eight parts (figure 2.1)

Fig. 2.2 The first representation of 3/8

then combine three of these parts (shaded); or alternatively combine three of the standard objects and then share the resulting combination into 8 parts (figure 2.2).

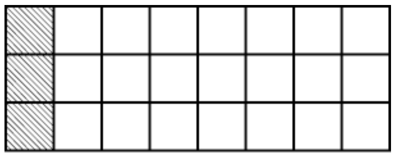

Fig. 2.3 The second representation of 3/8

In both cases we will end up with the three eighths of the standard object.

This is the closest example I have found in literature to what I call 'the same effect' of two different actions. It leads Skemp to the idea of representing sets of equivalent fractions as shown below (figure 2.3):

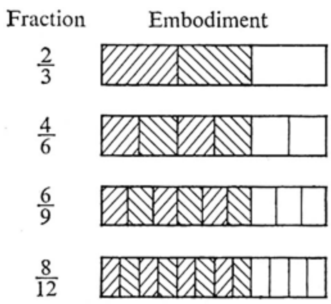

Fig. 2.4 representation of equivalent fractions

This suggests an alternative approach to the learning of fractions. Instead of speaking of the *mathematical* idea of 'equivalent fractions', it may be cognitively more appropriate to look at equivalent fractions as operations that 'have the same effect'.

In the case of vectors, we saw in chapter 1, figure 1.1 that it is possible to add vectors in different ways, and these ways have the same result. By embodying a vector through the action of translating an object on a flat table, we may focus attention on the fact that all the points on the object (or on the hand that moves the object) move in the same direction by the same amount. The shift of any such point can be represented as an arrow from its start point to its end point and any of these arrows is sufficient to represent the translation. As representations, they have the same effect. In this way we can give an embodied underpinning for the notion of free vector by focusing on the effect of a translation.

Lakoff writes:

> "Thought is embodied, that is, the structures used to put together our conceptual ideas grow out of bodily experience and make sense in terms of it; moreover, the core of our conceptual systems is directly grounded in perception, body movement, and experience of a physical and social character." (Lakoff, 1987, p. xiv)

With respect to symbols, however, Lakoff & Nunez (2000) say, "symbols are, just symbols, not ideas," and that the intellectual content of mathematics lies in its ideas for which symbols do the job of characterising their nature and structure. According to this viewpoint, abstract ideas make use of formulatable cognitive mechanisms, such as conceptual metaphors that import modes of reasoning from sensori-motor experiences.

My research is consistent with this statement as students often seem to 'know' the graphical symbol of an arrow representing a vector, without having a fully coherent understanding of ideas that give rise to its intellectual content. Students I have interviewed have had to be helped to attach a mathematical concept to the symbol of an arrow before they can manipulate it successfully in a full range of contexts, particularly in singular instances.

Lakoff & Nunez (2000) quote from cognitive science research, that most of our thought is unconscious and much of mathematical cognition happens at too low a level in the mind to be accessible. We draw conclusions from the world around us without being aware of it. We also have unconscious memory, which gives us implicit rather than explicit understanding. Schacter (1996) writes that the experiences we don't recall often have a detectable and measurable effect on how we behave. The theory of Lakoff & Nunez focuses on these unconscious mechanisms to suggest that understanding of mathematics uses the same cognitive mechanisms that are used for ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientation, basic manipulation of objects, iterated actions, and so on (pp. 27, 28).

On this basis it may be possible to reintroduce a concept, that causes a problem in developing understanding, so that, if we use the right experiences in the appropriate context, we may be able to set up the unconscious cognition in a more flexible manner, which will help the students in developing their knowledge. But how do we know when we introduce this idea to students that they build a proper concept and not just rote-learn and forget after a short time? The theory of embodiment suggests that we need to give appropriate experiences to underpin the concepts with bodily activity that integrates and supports the abstract ideas.

Socio-cultural theorists like Lave and Wenger (1991) view gaining knowledge as integration into a community of practice in which social actions are defined. For instance, students might be expected to learn the proper techniques of drawing using set-square, ruler and compasses. However, how does a community of practice pass on its more subtle conceptions that are carried out privately within our minds? Students may learn to perform mathematical manipulation of abstract symbols in accordance with the observed conventions, but there is still the question of the deeper conceptual meanings of the use of symbols to focus on the essential mathematical ideas free (as far as possible) from the coercive effects of specific embodiments.

2.3.5 The transition from embodiment to symbolism

The necessary shift from embodiment to symbolism has been detailed by Skemp (1971):

> First, we learn to manipulate concepts instead of real objects; then, having labelled the concepts, we manipulate the labels instead. Finally, perhaps, we reverse the process by re-attaching the concepts to the symbols and then re-embodying the concept in the real action with real objects from which they were first abstracted. (Skemp, 1971, p 83)

According to him we cannot use mathematics effortlessly unless we detach the symbols from their concepts and we have to be able to manipulate them without attention to their meaning. However he emphasises that this manipulation should be 'automatic' and not 'mechanical'. In automatic manipulation we can easily go back and reattach symbols to their meaning, while in mechanical manipulation the symbols stay meaningless. Skemp also says that:

> In mathematics, what we store is a combination of conceptual structures with associated symbols, and the former would therefore seem to be important for the retention of the whole.
>
> (Skemp, 1971, p. 85)

According to Hiebert and Lefevre (1986), symbols are viewed as cognitive aids, they "help to organize and operate on conceptual knowledge," (p. 15). They even go so far as to say that "The symbols can also produce conceptual knowledge," (p. 15). They further write that: "Being competent in mathematics involves knowing concepts, knowing symbols and procedures, and knowing how they are related,"(p.16).

Hiebert and Carpenter (1992) emphasise the importance of the symbolism to development of understanding and say that knowledge is represented internally, but communicating mathematics requires external representation:

> [...] when the relationships between internal representations of ideas are constructed, they produce networks of knowledge.
>
> (Hiebert and Carpenter, 1992, pp. 66-67)

They also say that students often make inappropriate connections or "represent information as isolated pieces," (p. 76) which cause difficulties in making sense of mathematical situations. Students build on prior knowledge and this may be procedural rather than conceptual, resulting at least in part from years of procedural and instrumental instruction.

Skemp (1979) describes a dynamic process of developing understanding: "to understand a concept, group of concepts, or symbols is to connect with an appropriate schema" (page 148), which puts the above theories into one sentence. However, this still begs the question of *how* the students connect all these bits of information into an appropriate schema.

To be able to conceive of ideas in a coherent form and to link them together requires a way of making this knowledge appropriate for comprehension and making connections. In particular, how do we put together embodied knowledge in a way which allows us to shift from embodiment to symbolism in a way that allows the symbolism to be used flexibly and meaningfully in a range of contexts?

2.4 Concept Images and Compression of Knowledge

Mathematical concepts are highly sophisticated mental constructions. Tall and Vinner (1981) define the *concept image* to be the total cognitive structure associated with a mathematical concept in an individual's mind. The ideas related to the given concept are continually constructed as the individual matures and are changing with new stimuli and experiences. Given such a range of cognitive structure, it is important to understand how the wider aspects of the concept image can be channelled into a thinkable entity that can be manipulated mentally in the mind.

Thurston (1990) described the way in which mathematical ideas start as a collection of disparate ideas which, through use and reflection, are compressed into easily recalled knowledge:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847)

On the same note, Crick (1994) states that the brain can make conscious decisions only by suppressing data and focusing on a limited quantity at a time.

Krutetskii (1976) writes:

> Retaining information in generalized and abbreviated form … does not load the brain with surplus information and thus permits it to be retained longer and used more easily. (Krutetskii, 1976, p. 300)

2.4.1 Cognitive Units

Barnard and Tall (1997) introduced the term *cognitive unit* for part of the concept image that can be held consciously in the focus of attention. A cognitive unit can be a symbol or representation or any other aspect related to the particular concept. For example in case of vectors it can be an arrow or a triangle of three vectors showing one side to be the sum of the vectors represented by the other twos. They hypothesise that powerful thinking arises through compressing information into rich cognitive units that can be manipulated in the mind.

> A powerful aspect of reflective thinking is the ability to compress a collection of cognitive units – which may be processes, sentences, objects, properties, sequences of logical deduction etc – into single entity that can be both manipulated as a concept and unpacked as a cognitive schema. (Barnard and Tall, 1997, p. 2)

This is particularly relevant to my own research as I seek ways of helping students to move from a range of experiences with the notion of vector to a central notion of free vector as a cognitive unit in its own right that has coherent meanings across a range of contexts.

2.4.2 Process-object encapsulation

A major theory that builds on the idea of internalising knowledge into thinkable entities is the APOS theory of Dubinsky and his colleagues, which is based on Piaget's epistemology of mathematics (Beth & Piaget, 1966). The acronym APOS stands for Action-Process-Object-Schema:

> An action is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. It may be a single step response, such as a physical reflex, or an act of recalling some fact from memory. It may also be a multi-step response, by then it has the characteristic that at each step, the next step is triggered by what has come before, rather

> than by the individual's conscious control of the transformation. … When the individual reflects upon an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes a process …. [Then] actions, processes and objects ... are organized into structures, which we refer to as schemas.
>
> (Cottrill, *et al.*, 1996, p. 171)

This theory is, in part, a theory of compression, from step-by-step actions to processes conceived as a whole that are then conceived as mental objects. In our approach to free vectors, this theory would suggest that individual actions (such as a shift of a triangle on a table) may be considered as a process (the transformation as a whole) and then conceived as an object (a free vector).

Sfard (1991, 1992) describes a similar sequence of compression:

> First there must be a process performed on already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three components of concept development will be called interiorization, condensation, and reification, respectively.(Sfard, 1992, pp. 64–65)

Though ideally the compression from action to process to object is highly desirable, Dubinsky and his colleagues found that college students often were able to move from action to process, but the next stage of producing a mental object was more difficult. (eg. Breidenbach, et al, 1992). They also reviewed their theory to explain that 'the construction of these various conceptions of a particular mathematical idea is more dialectic than a linear sequence' (Dubinsky and McDonald, 2001).

The serious question is therefore how a student can begin to think of a process as a mental object. A process occurs in time, an object is an entity that occurs in space (either real or imagined). Gray and Tall (1994) suggested that the mechanism by which this is done is through the use of a symbol to operate dually as process or concept. Thus the symbol 3+2 is both the process of addition and the concept of sum. They called a symbol that dually represents either *pro*cess or con*cept* a *procept*.

This highlighted the need for a symbol to function in a dual role, such as an arrow for a vector to represent both the process (as a movement from tail to nose) and the object (the arrow itself). However, the fact that an arrow has two distinct interpretations does not mean that students have a genuinely flexible view of vector. It seems that students can learn to operate with vectors as arrows in a limited way without constructing a flexible concept of a free vector. More insight is clearly required.

A clue is found in the description of Sfard: "First there must be a process performed *on already familiar objects*". The process-object encapsulation proposed by both Sfard and Dubinsky starts with actions on objects that already have meaning for the student. Tall and Gray (2001) suggest:

> [...] the theorised encapsulation (or reification) of a process as a mental object is often linked to a corresponding embodied configuration of the objects acted upon (which we henceforth refer to as *base objects*).
> (Gray & Tall, 2001, p. 266)

This idea links closely to Joshua's notion of *effect*. The compression of knowledge formulated in APOS theory does not begin with the A of 'Action' but with the B of 'Base object'. This gives a 'BAPOS' theory (proposed by Chae, 2002) in which Base objects are operated on by Actions, interiorized as Processes, encapsulated as Objects, within a wider Schema. By focusing on the *effect* of the Actions on the Base Objects, it now becomes possible to *see* the idea that represents the Process as a whole and can be symbolised as an Object. In the case of a translation, the base object is a figure (say a triangle) on a table and the effect is *the shift from the initial to the final position* of the base object without focusing on what happens in between. The effect can be represented by any arrow that has the same magnitude and direction as the shift, and any of these arrows represents the free vector that is the total shift from initial to final position.

This brings us closer to a possible theory of compression of the notion of translation into the concept of free vector. But we still need to seek a way in which this can be encouraged in our students. This takes us back to the fundamental idea of reflective thinking.

2.4.3 Reflection

Driver (1989) says that teaching involves the organisation of the classroom situation in a way which promotes learning outcomes.

Piaget (1985) suggests that one of the strategies to foster conceptual change is, to confront students with discrepant events and invoke a conceptual conflict which forces students to reflect on their conceptions as they try to resolve the conflict. However this can cause problems. Dreyfus, Jungwirth, and Elivitch (1990) found that their more able students react enthusiastically to conceptual conflict, but less successful students try to avoid the conflict or simply do not even recognise it.

As suggested by Barnard and Tall, if a student never builds a cognitive unit out of all the information he manages to assimilate then it would be very difficult for him to manipulate ideas and use them in solving problems presented to him.

Palmer & Flanagan (1995) found out that children develop their own ideas based on their own experiences. These ideas are often quite different from the accepted scientific viewpoints. Gilbert & Watts (1983) call them the “alternative conceptions” and Pines & West (1986) recognise that they significantly interfere with learning. One such concept is the Aristotelian idea that an action of continuous force keeps an object in motion. Sadanand & Kess (1990) found that 82% of senior high school students indicated that a force is required to maintain motion. Clement (1982) found that 75% of a group of university students indicated that there should be a force in the direction of the motion even after one semester of instructions in mechanics.

Kilpatrick (2002) suggests that students might have problems with understanding certain areas because they might not have encountered situations meaningful to them in which mathematics was important to know. Kilpatrick (2002) quotes the USA National Council of Teachers of Mathematics (1991), which specifies that the teacher’s role is to orchestrate the discourse so that the students in this class will function as an intellectual community. The teacher should set up a situation and then respond to what the students are saying by building on their observation, seeking clarification, and challenging them to explain and justify. This suggests that reflection is a process which would address these needs. The literature devoted to theories of how learners learn and how teachers teach (for example: Piaget, Skemp, Kilpatrick) have highlighted reflection as a central mechanism in thinking. This links closely with our earlier discussion of the constructivist approach to promoting learning advocated by Jaworski.

2.5 Bringing theories together

We are now moving to a position where the range of theories are bringing forward a general trend moving from initial intuitions from embodiment (which may include 'false intuitions') to a focus on the effects of actions to lead to symbolism. As we saw earlier, this is part of a cognitive development is described by Piaget in his stage theory of sensori-motor, pre-conceptual, concrete-operational and formal operational and by Bruner in his enactive, iconic and symbolic modes.

These were brought together by Biggs and Collis (1982) in their SOLO taxonomy to categorise the Structure of Observed Learning Outcomes. Biggs and Collis proposed five modes of cognitive development: *sensori-motor*, *ikonic*, *concrete-symbolic*, *formal* and *post formal*. They also note that, as each mode becomes available, it remains available alongside the new modes. Thus the introduction of the ikonic mode also includes the sensori-motor mode, which gives a combined embodied mode that encompasses both enactive and iconic (in the sense of Bruner). The concrete-symbolic mode includes the development of arithmetic and algebra and of the symbolic aspects of vectors. The formal modes include the notion of definition and deduction will not concern us here, but were suggested by Biggs and Collis (1982) to take the theory of Piaget beyond secondary education into graduate and postgraduate work.

These modes were consolidated into three by Gray and Tall (2001) in terms of embodiment (enactive and iconic), symbolic, and formal-axiomatic. In considering different types of mathematical concept they wrote:

> For several years [...] we have been homing in on three [...] distinct types of concept in mathematics. One is embodied object, as in geometry and graphs that begin with physical foundation and steadily develop more abstract mental picture through the subtle hierarchical use of language. Another is the symbolic procept which acts seamlessly to switch from a "mental concept to manipulate" to an often unconscious "process to carry out" using an appropriate cognitive algorithm. The third is an axiomatic object in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory. (Gray & Tall, 2001, p.70)

The three levels of object-construction described by Gray & Tall occur in the development of vectors, for instance, an arrow is an embodied object, the notion of the vector as a shift in space or as column vector has the structure of a procept and the axiomatic notion of vector space is an axiomatic object.

The research in this thesis inhabits the first two modes discussed here, the embodied mode which leads to graphical representations of vectors and the symbolic mode. To trace the development through the two modes, I again turned to the SOLO taxonomy where Biggs and Collis suggest that each mode has a common sequence of stages which can be used to test the quality of outcomes observed in tests designed for assessment. The stages are: *pre-structural* where no structure is used; *unistructural*, when student focuses only on a single aspect; *multi-structural,* when student focuses on several separate aspects; *relational* when the student relates different aspects together in a coherent way, and *extended abstract* where the student can see the concept from an overall viewpoint.

Bringing together a range of viewpoints, Pegg and Tall (2003) suggested that the SOLO theory encompasses a 'fundamental cycle' of conceptual development common to a range of distinct theories (figure 2.3).

	Davis	APOS	Gray & Tall	**Fundamental Cycle**	
Unistructural	VMS† Procedure		Base Object(s)	Base Object(s)	**Schema Building** ↓
		Action	Procedure	Isolated Actions Procedure	
Multistructural			[Multi-Procedure]	Multi-Procedure	
Relational	Process	Process	Process	Process	
Unistructural (Extended Abstract)	Entity	Object Schema	Procept	Entity Schema	

† VMS stands for Visually Moderated Sequence

Table 2.1 The fundamental cycle of conceptual construction

In each theory the first stage involves some kind of action on one or more base objects in which the focus of attention can be either on the object, or on the actions. Attention focused on the actions themselves can be consolidated into procedures (or multi-procedures) where there may be different ways (procedures) to carry out the same overall process. With support of symbols, students may at this stage construct a mental object as a cognitive unit which (according to the article) is both a schema within itself and also an entity that is manipulable within a wider schema of activities.

2.5.1 Combining modes

The previous section looked at the fundamental cycle of concept development that happens in a given mode. In our development we wish to see students construct the notion of free vector that relates across different modes. In the SOLO taxonomy, at the concrete symbolic stage, the student will also have available the embodied mode which may be viewed as a combination of enactive and iconic. As we shall see in the later study, some students may prefer to use the symbolic mode, others the embodied graphic mode and some will use a flexible combination of both.

2.5.2 Versatile thinking

Krutetskii (1976) identifies three basic types of mathematical cast of mind: the analytic type (who tends to think in verbal-logical terms), the geometric type (who tends to think in visual-pictorial terms), and the harmonic type (who combines characteristics of the other two)." (1976. p.xiv). He studied 'capable pupils' and discovered that a significant majority of them belonged to the third category. He suggests that such pupils are "quite ingenious in their visual interpretation of abstract relationships, but their visual images and schemes are subordinated to a verbal-logical analysis [...]. They are successful at implementing both an analytic and a pictorial-geometric approach to solving many problems," (Krutetskii, 1976, p. 326).

The distinction between different styles of thinking has long been a focus of attention in the literature. Brumby (1982), for example, noted two different strategies for solving a problem:

> (i) Immediately breaking a problem or task into its component parts, and studying them step by step, as discrete entities, in isolation from each other and their surroundings.
>
> (ii) An overall view, or seeing the topic/task as a whole, integrating and relating its various subcomponents, and seeing them in the context of their surroundings. (Brumby, 1982, p.244)

Her research suggested three distinct groups of students: those who consistently used only serialist/analytic strategies, those who used only global/holistic strategies, and those who used a combination of both, whom she described as *versatile learners*. Overall 42% of her sample maintained a serialist/analytic style, 8% were global/holistic and 50% were versatile.

In his thesis, following Brumby, Thomas (1988) used the term *versatile* to describe the complementary combination of global-holistic thinking and serial-sequential thinking. Subsequently it has been used to describe students who are able to use a variety of techniques in different contexts involving both linear procedural activities and also more flexible conceptual thinking (Blackett, 1990).

In this thesis we will describe students to be *versatile* if they are able to use their knowledge of free vector in a versatile way in solving problems in both the embodied and numeric modes.

2.6 Summary

In this chapter a range of theories of cognitive development have been considered from intuitive beginnings through instrumental and relational understand of procedural and conceptual knowledge. Although philosophers may regard intuition as a basis of all knowledge, it also depends on our human characteristics which can involve false physical intuitions at variance with subsequent theories.

Reviewing the theories of cognitive development proposed by Piaget and Bruner, I used the embodied theory of Lakoff to see the foundation of human development in embodiment (with links to Bruner's enactive and iconic modes) and focus on the transition to the symbolic mode, looking to constructivist theories to help students construct the shift from embodiment to symbolism, in flexible ways, in a variety of contexts. This involves the compression of knowledge from separate pieces of information into thinkable mental cognitive units.

Reviewing theories of Dubinsky, Sfard and Gray & Tall concerning the notion of 'process-object encapsulation', starting from step-by-step actions, interiorised to global processes and encapsulated as objects, we note the perceived difficulty of reconceptualising process as object.

At this point we introduce the notion of 'effect' that arose in discussion with the student Joshua in the Preliminary Study to use an extended BAPOS theory, in which Base Objects have Actions upon them, interiorising to Processes, then Objects, where the encapsulated object is now represented in terms of the 'effect' of the action on the base objects.

Introducing SOLO taxonomy not only incorporates the theories of Piaget and Bruner, but also has a cycle of concept construction that relates to theories of process-object encapsulation, to give a broader theory that can be used not only to describe the development of embodied and symbolic modes of operation, but also to relate them together in a versatile way.