Chapter 6

Pilot Study

6.1 Introduction

The pilot study tested:

- the design of questions to be used in the main study;
- the method of collecting and analysing both quantitative and qualitative data;
- whether the experimental lessons given to students have the anticipated impact on their results from the test.

However in the pilot study the test was only conducted twice, before the experimental lessons and after. In the main study the test was conducted three times and the focus was mainly on the difference between the pre-test and the delayed post-test because the long-term retention of knowledge was important in this analysis.

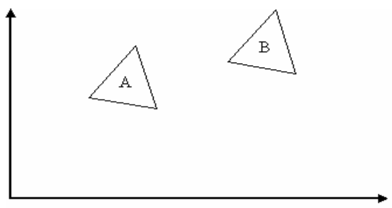
6.2 Design of the questions

The first two parts of the chapter show test questions and their intentions the next part shows how the stages of the cognitive development were awarded to students' responses; the last part shows the results of the pilot pre-test and post-test.

6.2.1 Cognitive development of vector

Figure 6.1 shows the diagram for the first question in the test.

1) In the picture the triangle has been translated from position *A* to position *B* as shown below:



(a) How can you represent the translation of the triangle?

(b) Can you draw a vector starting at the origin (0,0) which will represent the translation of the triangle from A to B? If so, show it on the drawing.

(c) Can you draw a vector not starting at the origin and not touching either of the triangles which will represent the translation from A to B? If so show it on the drawing.

Fig. 6.1 Test question 1

This question tested students' cognitive stage of vector awareness. The first part of question (a) expected students to represent a vector representing the movement of an object, either in a symbolic way, by representing vertical and horizontal components, or graphically as an arrow from one point on the object in position *A* to a corresponding point on the object at position *B*.

Students learn in their course that among all the equivalent vectors there is one which starts at the origin which is called the 'position vector'. Therefore, to test if they have developed so far in their cognitive understanding, the second part (b) expected a student to draw an equivalent vector to the one in part (a) but starting at the origin. The students who has continued on the cognitive development ladder to construct the idea of a free vector, are expected to understand that an equivalent vector can be represented anywhere on the page. To test this, part (c) of the question asked them to draw a third vector representing the translation but not touching the triangles or the origin.

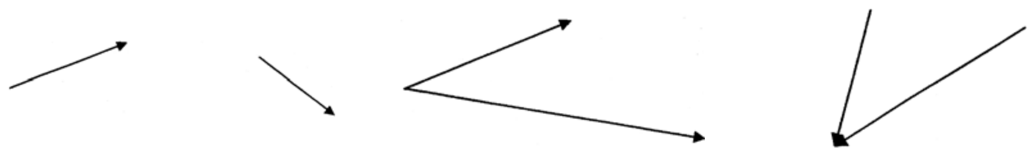
The first part gave students an opportunity to answer graphically or symbolically ("represent") but the second and third part asked them for the graphical representation ("draw"). Therefore symbolically this question may not encourage the higher stages of the cognitive development but the real purpose of this question is to see if students have a concept of a free vector, specifically if they can draw an arrow which is not on the triangle. For instance, students who conceive of the vector as a physical 'push' on the triangle might sense that for a vector to cause movement, it must actually touch the object being moved. They may believe this quite separately from the possibility that they are able to reproduce the learned response to draw a position vector at the origin. This question, despite the expectation of testing mainly the graphical mode was very important as it tests students' development from acting on the base object to the process of drawing equivalent vectors, to the concept of free vector.

6.2.2 Cognitive development of vector addition

The cognitive development of vector was further analysed in questions asking for vector addition.

The question presented in figure 6.2 asked students to add two vectors in the three situations shown below (a, b and c):

Question 2: In each case below add the two vectors:



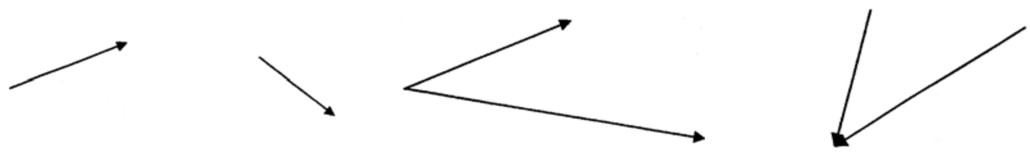
(a) **(b)** **(c)**

Fig. 6.2 Test question 2: adding vectors as arrows

This question includes three examples that have certain specific properties that are in some sense unusual. Part (a) is a prototypical example in mathematics when discussing addition of free vectors that are to be placed 'nose to tail'; the vectors are not touching or overlapping, so that either one can be shifted until its tail coincides with the nose of the other. However, from a physics viewpoint, this is not typical in the context of forces where the students are used to having the forces all acting on a single point. Part (b) is typical of two forces acting on a point, and from this viewpoint, it might evoke the use of the parallelogram law. However, as we have seen in the literature and in figure 2.1, it might evoke other meanings, such as two competing forces tugging at a point, or two sides of a triangle, leading to a misuse of the triangle law. Part (c) is considered to be a singular example that has not been discussed in Physics or Mathematics lessons. The manner in which the two vectors meet nose to nose may lead to misconceptions, such as the idea of two forces pressing on each other and perhaps cancelling each other out.

Question 3 (figure 6.3) is designed to test students' versatility by asking them for another way of adding vectors in question 2. They may respond by performing the sum in the same mode (perhaps nose to tail in figure (a) with the vectors in a different order) or in different modes (responding geometrically on one occasion and numerically on the other).

Question 3: If there is any other way you could have done any of the additions of the two vectors in Q2 show it here:



(a) (b) (c)

Fig. 6.3 Test question 3: adding vectors in another way

The next question is designed to test the students' understanding of free vectors in a more general case.

Question 4: Add the three vectors shown below:

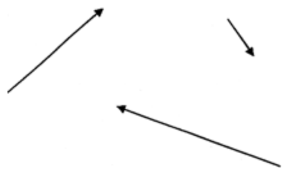


Fig. 6.4 Test question 4: Add three vectors

If they can add vectors in question 2 (a) or 3 (a), and also can add three vectors in question 4. (figure 6.4), then they are more likely to have some understanding of vector addition, even if they did not answer parts (b) and (c) in figures 6.2 and 6.3.

Students who learnt procedurally and not conceptually might only understand that adding two vectors will graphically be in a form of a triangle and not have a concept that whenever you add any number of vectors by shifting them 'nose to tail' you will get the resultant, which has the same *effect* as adding them all together (as shown later in figure 6.24(a)).

On the other hand they might realise the procedure of shifting vectors 'nose to tail' but not getting a triangle, they may not know what to do next (as shown later in figure 6.24(b)).

The next two questions are set up in two different physical contexts and are opened-ended:

Question 5: **Draw a representation of three forces and add them together.**

Question 6: **Draw a representation of two displacements and add them together.**

Students have an option, for example, to draw forces acting all in one direction or only in two, or all three forces acting from one point and then add them as ‘free vectors’. However if students are attached to the physical situation they might draw the forces, but unless they act in one dimension or maximum two (where they can use numerical methods) they may find them difficult to add.

In the case of displacements they might draw two vectors following each other and add them, however if they are confident with a concept of free vectors they might draw them separately and add them together.

The two questions above test how students operate when faced with vector addition in different physical contexts, and is included to see if it makes a difference to the stage at which they respond to vector addition.

The next question (figure 6.5) tests if the students can recognise answers in the midst of the drawn lines.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AB} + \overrightarrow{BD}$

(b) $\overrightarrow{DA} + \overrightarrow{ED}$

(c) $\overrightarrow{AB} + \overrightarrow{AE}$

(d) $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DE}$

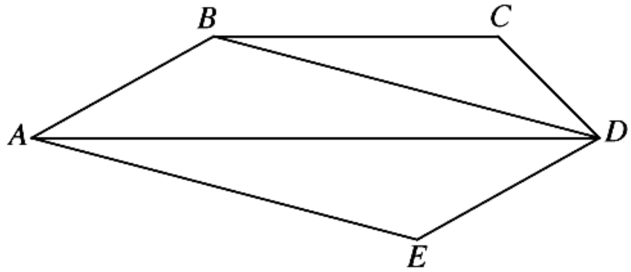


Fig. 6.5 Test question 7: adding vectors in a drawing

Questions like these are familiar to students from their Mathematics course in the previous year (Year 11). Part (b) can be answered in two ways, either using the commutativity law that $\overrightarrow{DA}+\overrightarrow{ED}=\overrightarrow{ED}+\overrightarrow{DA}$ which, in turn equals $\overrightarrow{EA}$, or by seeing that $\overrightarrow{AB}$ is parallel to $\overrightarrow{ED}$ and has the same magnitude, and using the idea of equivalent free vectors to give the answer $\overrightarrow{DB}$. Because of the involvement of the commutative law of addition part (b) of the question has been categorised as a 'singular' case.

The last set of questions (figure 6.6) has been assigned as a 'singular' case. The students were not familiar with the answers not being part of the diagram and it was considered that in order to answer them, they had to be familiar and confident with the idea of free vector.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AD}$ and $\overrightarrow{CD}$

(b) $\overrightarrow{AD}$ and $\overrightarrow{BC}$

(c) $\overrightarrow{AC}$ and $\overrightarrow{BD}$

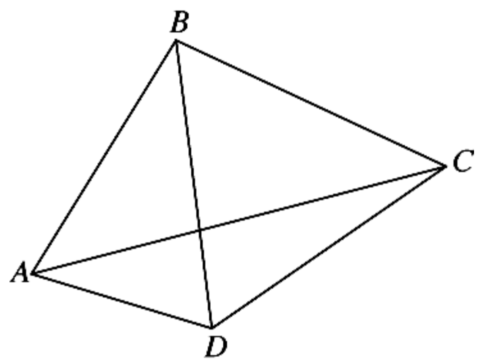


Fig. 6.6 Test question 8: more sophisticated addition in a drawing

Part (a) of question 8 is singular for two reasons: one because the answer is not part of the diagram (which was always the case in the students' earlier experience); but also because two vectors meet at one point (as in questions 2c and 3c). Part (c) of question 8 is also singular for two reasons, first that the result is not part of the diagram and the second that two vector cross each other. Students who can answer all questions in figure 6.6 are therefore considered to be at the top of the ladder indicating their cognitive development in at least one of the two modes (graphical or symbolic).

The pilot study tests the final design of questions and the way the responses were going to be evaluated in the main study. In the following two sections it can be seen how the responses to the questions will be analysed.

6.3 Method of collecting quantitative data

In the main study the test was given to students 3 times, before the course, straight after the course (which included the experimental lessons in case of experimental group) and the delayed post-test given to students half a year after the course. However in the pilot study the test was given only twice, before the course and straight after the course which also involved two experimental lessons for one group out of two taking part.

The quantitative data analysis focused on the stages attained by students in the graphical and symbolic modes of operation, as formulated in chapter 4, figure 4.6. In practice, the students often responded in ways that required careful analysis to place them in appropriate stages. This was done with help of another teacher from the same school who taught Pure Mathematics with Mechanics for many years but was not involved at any stage with the students under investigation. The researcher and the teacher independently allocated stages to a sample of nine varied responses from students to the test. These allocations were discussed and final versions established. Thereafter the rest of the responses were allocated stages of cognitive development according to the agreed format.

The principle was to give the highest stage for each question, consistent with the response, if there was no graphical or symbolic response at all, then stage 0 was awarded, even though the stage 0 is also for the intuitive responses. In both cases it meant that student did not reach the first stage of the cognitive development ladder. The more precise interpretation of such results had to be tested through the interviews.

It has been also decided that in the symbolic mode the answers given in letters or numbers will be treated the same. It might be debated if those two responses show different cognitive level of development, but it will not be part of this study. The student had to satisfy a certain stage at least twice (in two different questions or parts of the question) before being awarded the stage in the general case (taking into consideration all the test questions) but only once in specific questions analyses ('singular' cases, different contexts cases).

As could be seen from the questions described in part 1 of this chapter, some were more suitable for graphical mode responses than others and question 1 (b) and (c) specifically asked for the graphical responses, which could have influenced some students that this was expected mode for the rest of the test. However the overall expectation was that students may answer in the mode they are more familiar with in most of the questions and perhaps show the other ability either when their favoured mode is not possible, or if they are asked to do it differently. A sample of students answering in only one mode throughout the test was interviewed to check if in fact only one mode was familiar to them.

For the reasons stated in the previous paragraph, when making a judgement for overall stages of development, the final analysis was performed in two ways. The first way used all the responses given by the student to prescribe an overall stage of development. There were 17 questions in total and the student had to achieve their highest stage twice to be given it. If student answered, for example, once at stage 4 and once or more at stage 3 then stage 3 was given. In the case where a student answered once at stage 4 and once or more at stage 2 then stage 3 was given. In general the rule is to take the two highest stages awarded, calculate the average, and round it down to the nearest whole number.

The second way involved focusing on all the questions which contributed to a specific aspect of study. There were two cases considered: the 'singular' cases (four questions) and the questions testing different physical contexts (displacements and forces) of which there were two. Because of the smaller number of such questions, students had to gain the stage only once to be prescribed that overall stage in the aspect under consideration.

The pilot study only looks at the overall stages gained by students, however the main study also will analyse the specific type of questions as the changes occurring in the students' responses show the more precise insight into their development.

The stages gained by each student in both graphical and symbolic modes were then plotted on a scatter graph. The scatter graph was divided into 25 regions and these regions were given categories, developed in chapter 4, as shown in table 6.1.

This begins with stage 0, in which students responded essentially only in terms of physical intuition without any clear evidence of mathematical activity. Such a response in both graphic and numeric modes was classified as ***physical intuitive***. The next identifiable level occurs in a way that focuses on mainly symbolic or mainly graphical representations at lower stages of cognitive development. I took the decision to assign performances that attained level 1 in one of the modes but failed to reach level 2 in the other as being ***uni-modal***. This was subdivided into ***lower uni-modal*** if the activities in the higher scoring mode was at stage 1 or 2 and ***higher uni-modal*** if at stage 3 or 4. If both modes reached level 2, then the performance was categorized as ***multi-skilled***. Performances reaching at least level 3 in both modes are classified as ***versatile*** and those who attain level 4 in both modes are termed ***fully integrated***.

graphical mode	stage 4	higher uni-modal	higher uni-modal	multi-skilled	versatile	fully integrated
	stage 3	higher uni-modal	higher uni-modal	multi-skilled	versatile	versatile
	stage 2	uni-modal	uni-modal	multi-skilled	multi-skilled	multi-skilled
	stage 1	uni-modal	uni-modal	uni-modal	higher uni-modal	higher uni-modal
	stage 0	intuitive	uni-modal	uni-modal	higher uni-modal	higher uni-modal
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Table 6.1 Table of the second stage of the categorisation

The data in the main study is going to be presented in the form of the table above but instead of names of the categories there will be indication of how many students in each group responded in those categories. The sign 'A' will indicate a student from group A and the sign 'B' will indicate a student from group B. The χ^2 test will show if there is a significant difference between the control and the experimental group by comparing the number of students in two regions marked in a thicker line: one including intuitive and uni-modal categories; another including higher uni-modal, multi-skilled, versatile, and fully integrated.

6.3.1 Quantitative Data Analysis of Understanding the Symbol of a Vector

Figures 6.7 and 6.8 (taken from figure 4.6) show how the stages will be allocated to students' test responses as far as their cognitive development in understanding the concept of vector is concerned graphically and symbolically.

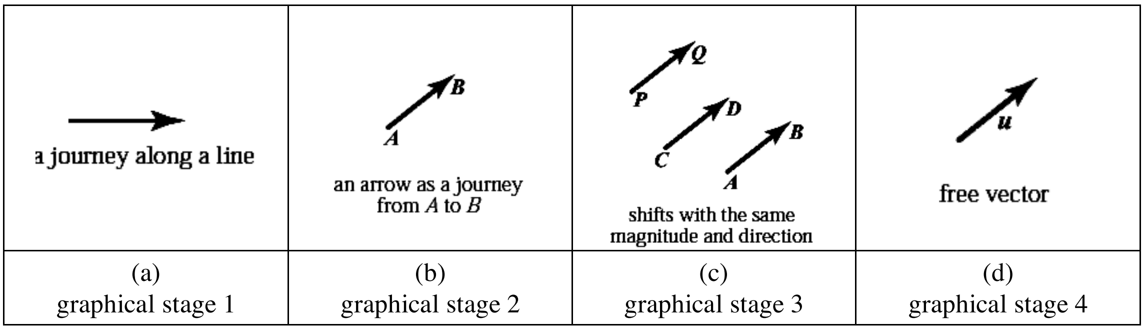


Fig. 6.7 Four stages of cognitive development of vector in the graphical mode

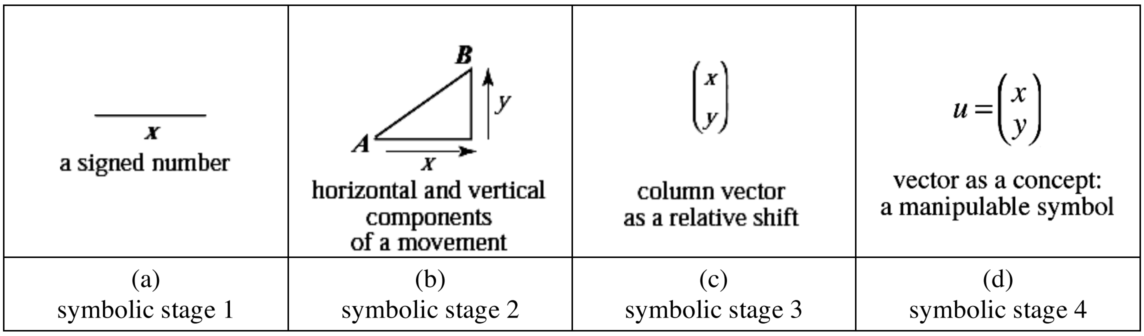


Fig. 6.8 Four stages of cognitive development of vector in the symbolic mode

In addition, the relationship between the stages of development in the symbolic and graphic responses will be categorised using the corresponding cycles shown in figure 4.8, which were formulated in chapter 4 in the following terms:

physical-intuitive: signifies the performance of those students who do not have any specific understanding of the graphical or symbolic representation of a vector;

uni-modal: applies to the students who can operate in basically only at stages 1 and 2 in both modes (symbolic or graphical);

higher uni-modal applies to the students who can operate in basically only one mode (symbolic or graphical) at stages 3 or 4 but only at stage 0 or one at the other mode;

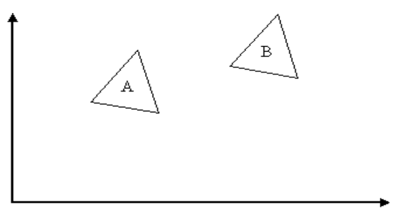
multi-skilled: students who show that they can use both modes of vector representation but do not use them flexibly (the context affects the level of their responses);

versatile: students who use both modes of operation flexibly whatever the context.

fully integrated: relates to the students who recognise the concept of free vector and see it as a mathematical manipulable symbol whatever the context and using the appropriate mode of representation (graphical/symbolic: numerical and algebraic).

The first question is repeated in figure 6.9, so that it can be compared with the responses in figure 6.10. The preliminary study indicated that students often understand the *position vector* (vector starting at the origin) as a movement of an object. Similarly students often showed a translation as an arrow from a specific point on an object to the corresponding point on its translated image.

1) In the picture the triangle has been translated from position *A* to position *B* as shown below:



(a) How can you represent the translation of the triangle?
(b) Can you draw a vector starting at the origin (0,0) which will represent the translation of the triangle from A to B? If so, show it on the drawing.
(c)Can you draw a vector not starting at the origin and not touching either of the triangles which will represent the translation from A to B? If so show it on the drawing.

Fig. 6.9 Test question 1

The questionnaire reveals only the responses written at the time and do not necessarily reveal whether the students have possibly broader levels of flexibility available to them beyond the written answers. This possibility will be considered in the qualitative analysis.

Figure 6.10 show six examples of students' responses.

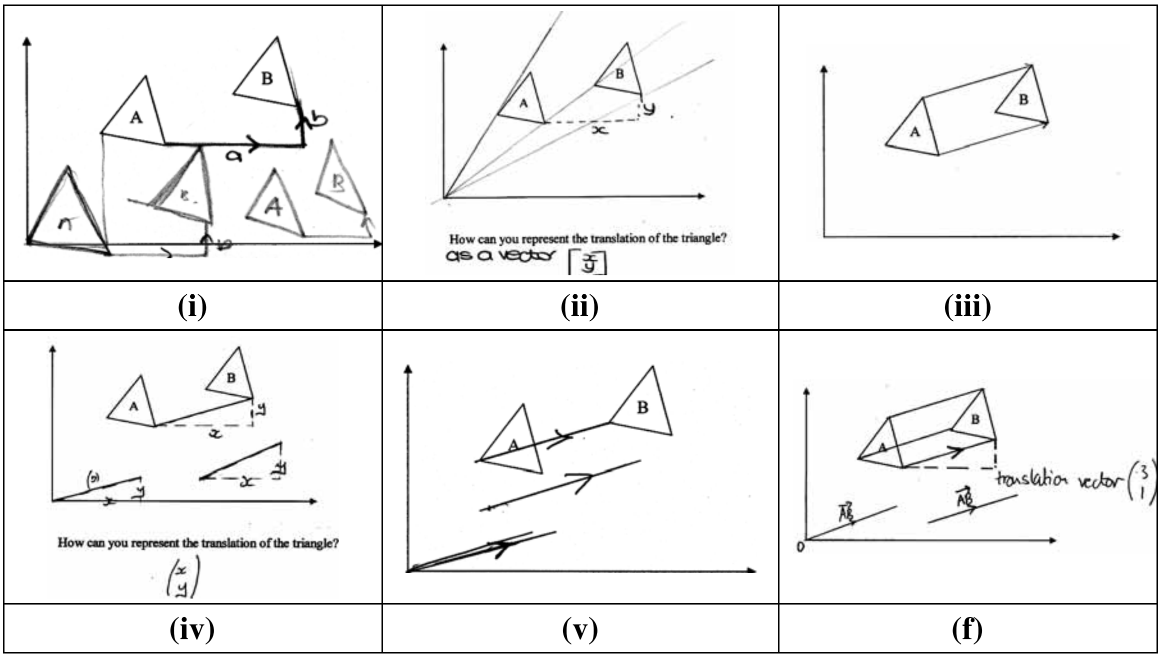


Fig. 6.10 Examples of students' responses to test question 1

The categorisation of students' responses, at different stages of the cognitive development of a vector concept, according to the examples shown in figure 6.4, is discussed below.

The student who responded as shown in figure 6.10 (i) was categorised to be at the stage 0 of the graphical representation but at stage 2 of the symbolic representation of vector. (S)he presented the translation symbolically as horizontal and vertical components but graphically only translated an object without showing the action as an arrow from one point to another. This student is also at the stage of action on an object and does not use the symbol of a vector (an arrow) to indicate the translation. (S)he does not realise the equivalence of vectors but only the equivalence of movements.

The student who responded as shown in figure 6.10 (ii) was categorised to be at stage 0 for the graphical representation as there are no arrows on the drawing or even an indication of moving lines parallel to each other. However the student was given stage 3 for the symbolic representation as (s)he not only showed the translation as the horizontal and vertical movement but also as a column vector, as a relative shift. This student's written response shows no awareness of the notion of vector in a graphical form as even the *x* and *y* components have no arrows on them.

The student who responded as shown in figure 6.10 (iii) responded at stage 3 for the graphical representation as the two arrows represent a 'journey' of the object from a specific point to another specific point and a shift with the same magnitude and direction. However the student did not respond symbolically and from the principles established earlier was given stage 0 in that mode.

The student who responded as shown in figure 6.10 (iv) was categorised to be at stage 3 of the graphical representation and at stage 3 for the symbolic representation. Although the translation is only represented as a line (stage 1) the student shows the concept of 'the same magnitude' and to some extent 'the same direction' by placing *x* and *y* in the same order and revealing some indication of the direction.

The student who responded as shown in figure 6.10 (v) responded at stage 4 for the graphical representation as (s)he drew the notion of free vector, not attached to the object or any specific point, however, (s)he did not respond symbolically and was given stage 0.

The student who responded as shown in figure 6.10 (vi) was categorised to be at stage 4 for the graphical representation as the notion of free vector is indicated graphically and stage 3 for the symbolic representation as (s)he showed a column vector as a relative shift.

Some students did not give any symbolic response to question in figure 6.9 and the data about their cognitive development in the symbolic mode had to be collected from other questions. From the preliminary study it seems that the changes in the symbolic mode are not statistically significant after the experimental lessons, mainly due to the fact that the underlying data is not clear. The interviews tend to reveal more information; it is here that more insight appears, though not for all students.

The second example of the questions given to students is shown in figure 6.11.

2 (b) add the two vectors together

3 (b) If there is any other way you could have done any of the additions of the two vectors in Q2 show it.

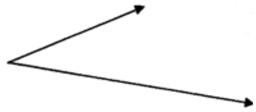


Fig. 6.11 Questions 2(b) and 3(b).

Figure 6.12 shows the examples of the responses of four selected students. The top picture shows the response to question 2(b) and the bottom picture response of the same student to 3(b).

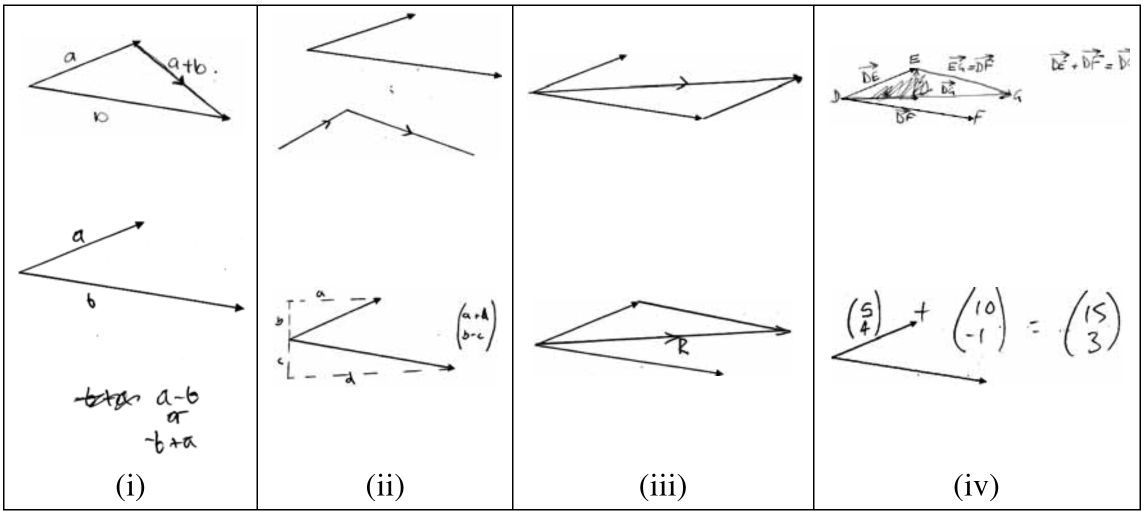


Fig. 6.12 Examples of four students' responses to questions 2(b), 3(b)

The response in part (i) of figure 6.12 indicates that in a graphical mode a student might interpret vectors as a journey along the line as vectors follow each other in the top answer, described as stage 1. On the other hand, on the basis of the preliminary and pilot studies, some students might be treating **a** and **b** as the position vectors of some point *A* and another point *B* and draw a displacement vector from *A* to *B*, which would indicate stage 2. On this occasion the student wrote first the expression **-b+a** below which he crossed and changed to **a-b**. This would imply that the student was considering a journey, (along **b** in the reverse direction, then along **a**)and therefore the first interpretation was assumed to be the more likely, and the graphical response was categorised at stage 1. The symbolic response was categorised as stage 1 because student only assigned letters to the vectors and did not try to manipulate symbols in any meaningful way.

In part (ii) the student shows that (s)he can shift vectors with the same magnitude and direction but does not show the resultant. The student therefore does not recognise the full idea of the same *effect*. The student was allocated stage 3 for this graphical response. The symbolic response which the student gave to 3(b) question was awarded stage 3, as this indicated the column vector as a representation of the relative shift.

The student in part (iii) of figure 6.12 was awarded stage 4 for the graphical response. This student shows not only an understanding of the concept of free vector but also the concept of the commutative law of addition.

Part (iv) of figure 6.12 shows a student who was allocated stage 4 for the graphical response. This student not only can shift vectors with the same magnitude and direction but also can add them showing concept of *effect*. This student has been also awarded stage 4 for the symbolic representation as (s)he uses a column vector as a manipulable symbol.

The questions discussed above use the concept of vectors formulated in general situations but for someone who can only think about the vectors as symbols related to the physical world, they could be interpreted, for example, as a displacement in case of question 1 (figure 6.9) and forces in case of question 2 and 3 (figure 6.11). These types of question we call *generic cases* (chapter 1, p. 4). However the students are given examples of vectors whose 'noses' meet at a point or where vectors cross we call them *singular cases* (questions which might cause confusion from the physical/intuition point of view). It therefore seemed important to show how responses to such questions were awarded with stages.

The example of questions which were categorised as 'singular' cases are shown in figures 6.13 and 6.15 and are discussed next.

Add the two vectors

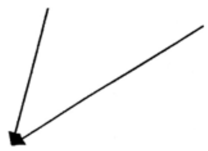


Fig. 6.13 Singular question

Figure 6.13 shows two vectors which meet at one point. According to the teachers and the preliminary tests this example goes against many students' intuition.

The example of the way different stages were awarded to students' responses is shown in figure 6.14. Parts (i)-(iii) show graphical responses and parts (iv)-(vi) show symbolic responses.

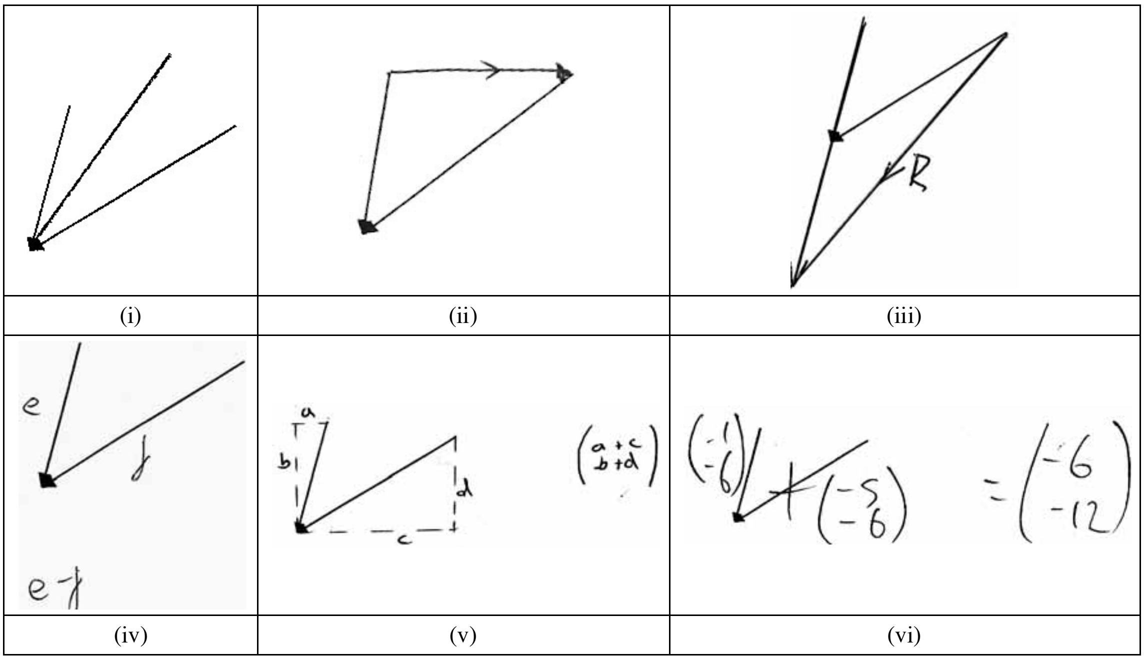


Fig. 6.14 Allocation of stages to the responses to the singular case in 2(c), 3(c)

Part (i) shows what seems to be an intuitive response. The student seems to be aware in which more or less direction the resultant should be, however neither the magnitude nor direction of the resultant are correct and therefore stage 0 was given.

In part (ii) It is not clear if students shows some sort of intuitive response or (s)he is trying to 'close a triangle' from the beginning of one vector to another. As different interpretations give the highest stage 2 for the graphical response then according to the general principle adopted for the ambiguous cases, stage 2 was given to the response.

Stage 4 was given for the response in part (iii) as student shifted one vector with the same magnitude and direction and showed the resultant.

Part (iv-vi) responses were given stage 0 for the graphical response.

Part (iv) was given stage 1 for the symbolic response as the student simply put a signed letter to the arrow.

Parts (v) and (vi) are similar in the final response although it can be debated if using letters or numbers shows a difference in the stage of development. However as decided at the beginning of this section of the chapter, it is not part of the analysis. There is however slight difference in the way two responses are presented in part (v) student shows column vector as a relative shift with horizontal and vertical components being added, however in part (vi) it is evident that two vectors were added to show the answer and therefore higher stage was given to part (vi) – stage 4, than to part (v) – stage 3.

Another 'singular' case set of questions is shown in figure 6.14.

Using the drawing below, or otherwise, add:

(a) $\overrightarrow{AD}$ and $\overrightarrow{CD}$

(b) $\overrightarrow{AD}$ and $\overrightarrow{BC}$

(c) $\overrightarrow{AC}$ and $\overrightarrow{BD}$

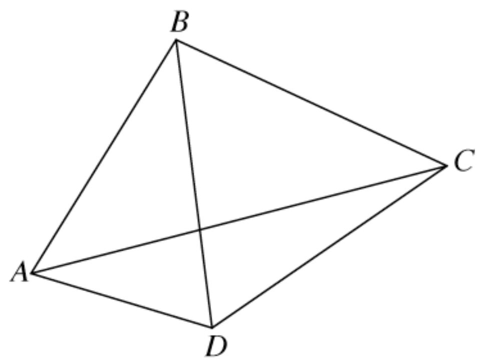
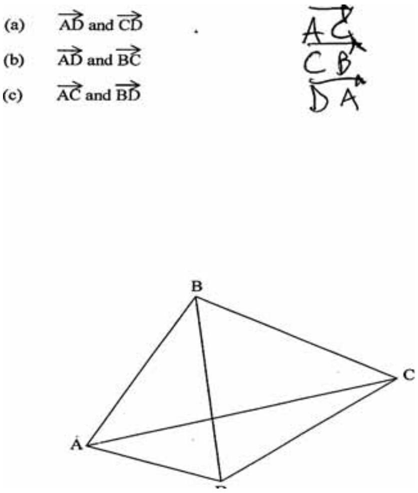


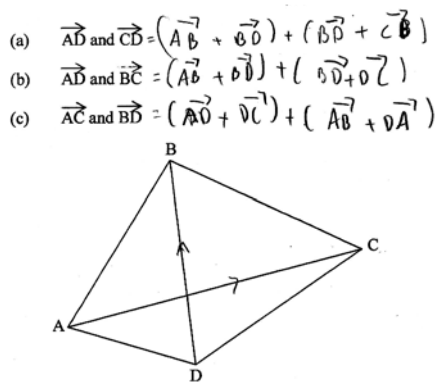
Fig. 6.15 Singular question

The question shown in figure 6.15 is set up differently from the questions students met in the previous year in their text-book. The questions they were dealing before had all answers as part of the diagram. In this case none of the answers fit that pattern. Part (a) of this question has also two vectors meeting at one point and part (c) has two vectors crossing each other.

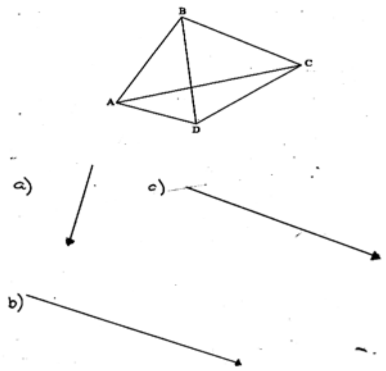
Figure 6.16 shows a selection of students' responses to the above and allocation of stages to these responses.



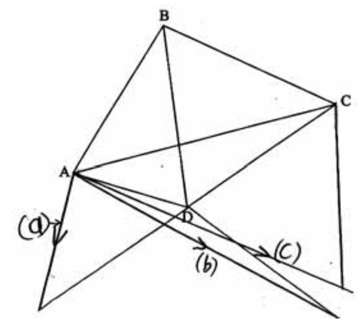
(i)



(ii)



(iii)



(iv)

Fig. 6.16 Examples of students' responses to singular questions.

In figure 6.16 part (i), the student responded similarly to the response in figure 6.14 part (ii) and therefore stage 2 in graphical mode was awarded. The student gave a vector response (with an arrow above the letters) to the answers and therefore stage 1 was given for the symbolic response.

In figure 6.16 part (ii) the student drew the arrows along the lines AC and DB and therefore stage 2 (describing an arrow as a journey from one point to another) was awarded. As in part (i), the student gave a vector response to the arrows and therefore stage 1 was given in the symbolic mode.

In figure 6.16 (iii) the graphical responses were given stage 4 as the student shows the correct resultants as free vectors. As there is no evidence of symbolic addition, stage 0 was assigned to the symbolic mode.

In figure 6.16 (iv), in the graphical mode the student shifted vectors with the same magnitude and direction and also manipulated them to perform the correct addition, therefore stage 4 was given despite the lack of arrows on equivalent vectors. However there is no indication of any symbolic use of vectors and therefore stage 0 was given for that mode.

6.3.2 Quantitative Data Analysis of Understanding Vector Addition.

The figures 6.17 and 6.18 show the theory developed in chapter 4, how the stages were going to be allocated to students' test responses as far as their cognitive development in understanding vector addition is concerned,

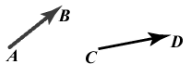
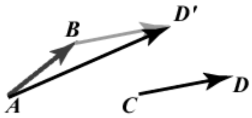
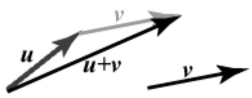
 add arrows	 can only add arrows as journeys if *B* is coincident with *C*	 $AB + CD = AD'$ using triangle (or parallelogram)	 adding ***u*** and ***v*** using triangle (or parallelogram)
graphical stage 1	graphical stage 2	graphical stage 3	graphical stage 4

Fig. 6.17 Stages of cognitive development of vector addition in the graphical mode

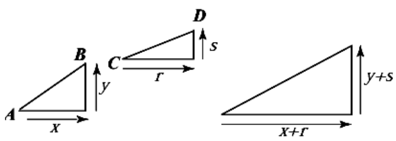
$\frac{x \quad r}{x + r}$ add numbers	 adding components (eg by counting squares)	$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} x+r \\ y+s \end{pmatrix}$ adding vectors by adding components	$u = \begin{pmatrix} x \\ y \end{pmatrix}$ $v = \begin{pmatrix} r \\ s \end{pmatrix}$ $u + v = \begin{pmatrix} x+r \\ y+s \end{pmatrix}$ Adding vectors
symbolic stage 1	symbolic stage 2	symbolic stage 3	symbolic stage 4

Fig. 6.18 Stages of cognitive development of vector addition in the symbolic mode

These stages as in case of concept of vector can be plotted on the scatter graph as shown in table 6.1.

The second level of categories come from the scatter graph presented in the table 6.1 depends on the stages students were awarded initially in the tests. The description of the categories give some idea of the student that fits into them.

physical-intuitive: signifies the performance of those students who do not have any specific understanding of vector addition in a graphical mode and at the same time do not present addition symbolically;

uni-modal: applies to the students who can operate in only at stage 1 or 2 in either mode (symbolic or graphical;

higher uni-modal applies to the students who can operate in basically only one mode (symbolic or graphical) at stages 3 or 4 but only at stage 0 or 1 at the other mode;

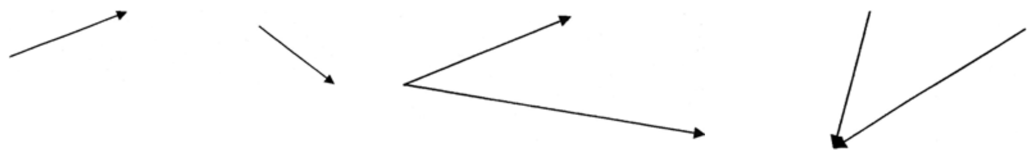
multi-skilled: students who show that they can use both modes in vector addition but do not use them flexibly (the context affects the level of their responses);

versatile: students who use both modes of operation flexibly whatever the context.

fully integrated: relates to the students who recognise the concept of free vector in vector addition whatever the context and using the appropriate mode of representation (graphical/symbolic: numerical and algebraic).

The students' responses to 7 different sets of questions on addition of vectors were considered in this part of the analysis. The first two questions (repeated from figures 6.2 and 6.3) are shown in figures 6.19 and 6.20 and the analysis of a sample of students' responses follows.

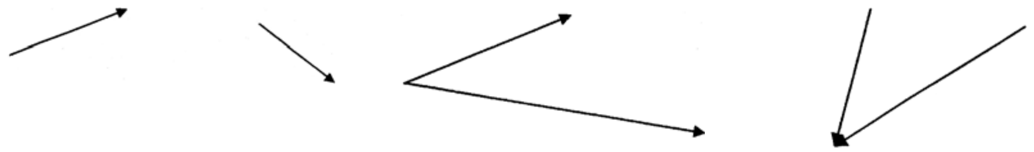
Question 2: In each case below add the two vectors:



(a) **(b)** **(c)**

Fig. 6.19 Test question 2

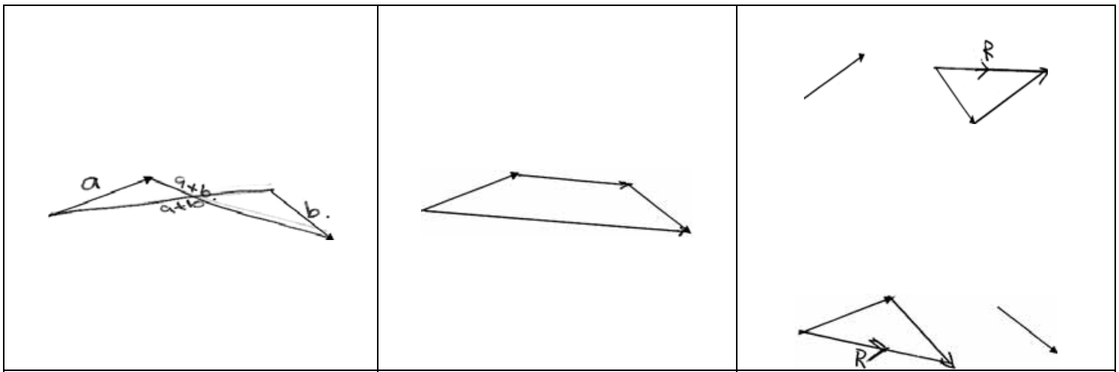
Question 3: If there is any other way you could have done any of the additions of the two vectors in Q2 show it here:



(a) (b) (c)

Fig. 6.20 Test question 3

The examples of the graphical responses, to questions 2 (a) and 3 (a) are presented in figure 6.21.



(i) (ii) (iii)

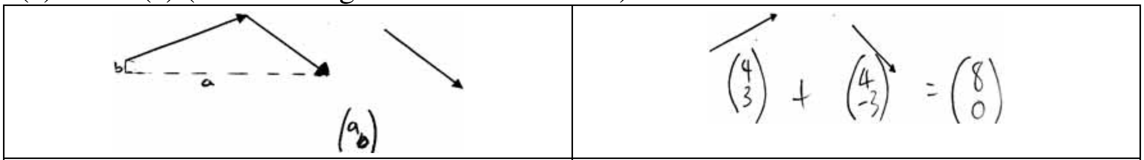
Fig. 6.21 Example of graphical responses to questions 2 (a) and 3 (a)

The response in part (i) of figure 6.21 shows no graphical rules of addition applied at all, and therefore the student was given stage 0 (however in the symbolic mode the student was given stage 1 as the letters representing vectors were added).

In part (ii) the student added an additional vector from the nose of the first vector to the tail of the second, but then seemed to go on to add all these three vectors together to complete the polygon. He could be only given stage 0, as he created his own continuity of journey by inserting the extra arrow.

Part (iii) of figure 6.21 show responses from the same student to questions 2 (a) and 3 (a) respectively. The student seems to have knowledge of the commutative law of addition and therefore is assumed to realise the concept of free vector and is given stage 4.

Figure 6.22 shows two examples of typical symbolic responses to questions 2 (a) and 3 (a) (shown in figures 6.19a and 6.20a).

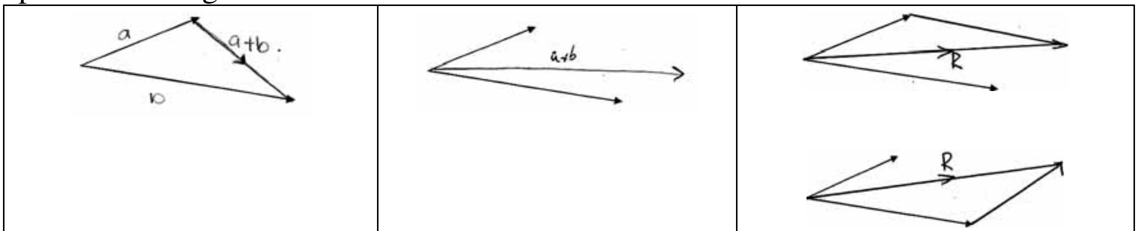


(i)	(ii)

Fig. 6.22 Examples of graphical responses to questions 2(a) and 3 (a)

The examples shown in figure 6.22 (i) was given stage 3 in the graphical mode as the student added arrows as a journey, and stage 3 in the symbolic mode as the student shows the resultant in the form of vertical and horizontal components only. The response in figure 6.22 (ii) was given stage 3 as student added vectors by adding components.

The examples of the graphical responses, to questions 2 (b) and 3 (b) are presented in figure 6.23.



(i)	(ii)	(iii)

Fig. 6.23 Example of graphical responses to questions 2 (b) and 3 (b)

The response in figure 6.23 (i) was given stage 0 because student put the symbol **a+b** on the arrow drawn as the answer. If (s)he had not done this, it may be considered that the student assumes that the arrow is the resultant of addition of the other two vectors. However by writing **a+b**, the student seems to indicate that this is a resultant. From the experience in the preliminary study this happens when a student thinks that the resultant will go in the direction of the longer arrow (force), which is an intuitive response.

The response in figure 6.23 (ii) seems as if the student used the parallelogram rule but only approximately, however, according to the principle of giving the highest mark, stage 4 was given. The responses in figure 6.23 (iii) come from the same student and were given stage 4. This student seems to have the concept of free vector and uses the commutative law of addition in the graphical mode.

Figure 6.24 shows two examples of typical symbolic responses to questions 2 (b) and 3 (b) (shown in figures 6.19a and 6.20a).

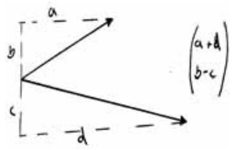
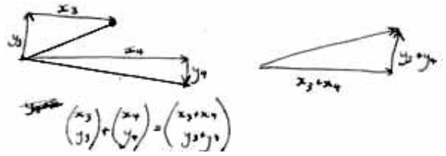
	
(i)	(ii)

Fig. 6.24 Examples of symbolic responses to questions 2(b) and 3 (b)

Both responses shown in figure 6.24 were given stage 3 in the symbolic mode. Additionally response shown in figure 6.24 part (ii) was also given stage 3 in the graphical mode, as student seem to be using the triangle addition of the components.

The examples of the graphical responses, to questions 2 (c) and 3 (c) are presented in figure 6.25.

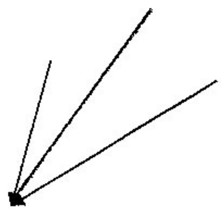
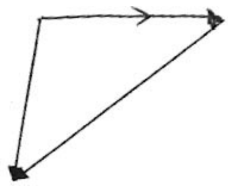
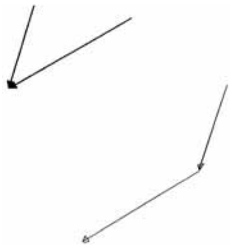
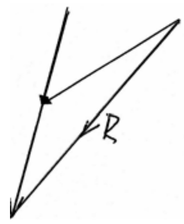
			
(i)	(ii)	(iii)	(iv)

Fig. 6.25 Example of graphical responses to questions 2 (c) and 3 (c)

The responses in figure 6.25 refer to the *singular case* presented in figures 6.19 (c) and 6.20 (c). In part (i) the arrow is too short for the parallelogram law to have been used and the direction is only approximate, that is why the intuitive response has been given in the form of stage 0. The response in part (ii) was also given stage 0. In figure 6.25 part (iii) the student joins the vectors ‘nose to tail’ but does not add them which is considered to be stage 2 (adding arrows as a journey). The response in part (iv) was given stage 3 as a single answer, however, if the same student were to show an understanding of the commutative law in question 2 (c) and 3 (c), then stage 4 would be given.

Figure 6.26 shows two examples of typical symbolic responses to questions 2 (c) and 3 (c) (shown in figures 6.19a and 6.20a).

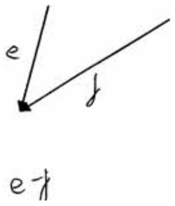
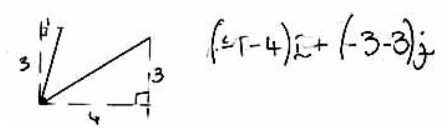
	
(i)	(ii)

Fig. 6.26 Examples of graphical responses to questions 2(c) and 3 (c)

The first response (figure 6.26 (i)) was given stage 0 as student as the student created the continuity by changing the sign of one of the vectors and therefore thinks of a journey. This seems to be a symbolic equivalence to the graphical answer shown in figure 6.25 (ii).

The second response in figure 6.26 (ii), was given stage 3 as the resultant is shown in a vector form obtained by adding the components.

The preliminary study gave some indication that students might be graphically adding two vectors procedurally in a way that the triangle has to be obtained and might have tried to apply this procedure to addition of three vectors trying to make a triangle, or being unable to draw the resultant if the three vectors did not make a triangle. Question 4 shown in figure 6.27 was given to students for that reason.

Question 4: Add the three vectors shown below:

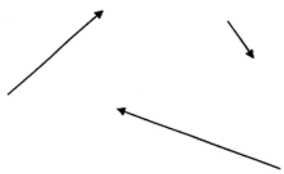


Fig. 6.27 Test question 4: Add three vectors

The examples of the responses to question 4 are shown in figure 6.28.

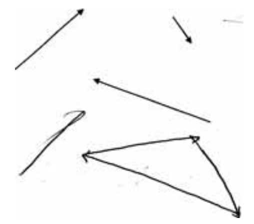
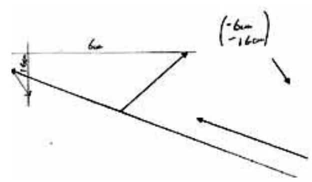
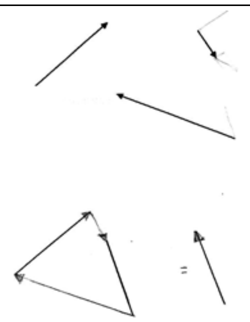
	
(i)	(ii)
	
(iii)	(iv)

Fig. 6.28 Response to question 4: Add three vectors

Part (i) of figure 6.28 shows a response described earlier in which the student might recall the idea of two vectors being added using the triangular law and attempt to force the three vectors displayed to make it look like a triangle. The student stretched and shrank some of the arrows and distorted angles so that the three sketched vectors make a triangle. This response was given stage 0 in the graphical mode; it was also given stage 0 in the symbolic mode as no symbols were used.

Part (ii) was given stage 2 in the graphical mode as the student placed arrows together ‘nose to tail’ like a journey but did not add them. Again stage 0 was given for the symbolic response or rather lack of it.

Part (iii) response was given stage 3 for the graphical response as the student shifted the vectors ‘nose to tail’, and stage 3 in the symbolic mode as the student shows the resultant in the form of the horizontal and vertical components.

Part (iv) was awarded stage 4 in the graphical mode and stage 0 in the symbolic mode.

The student not only added 3 vectors but also shows the resultant again as a free vector.

The different physical contexts questions are also going to be analysed separately in the main study.

Question 5: **Draw a representation of three forces and add them together.**

Question 6: **Draw a representation of two displacements and add them together.**

The examples of the responses to question 5 questions are shown in figure 6.29.

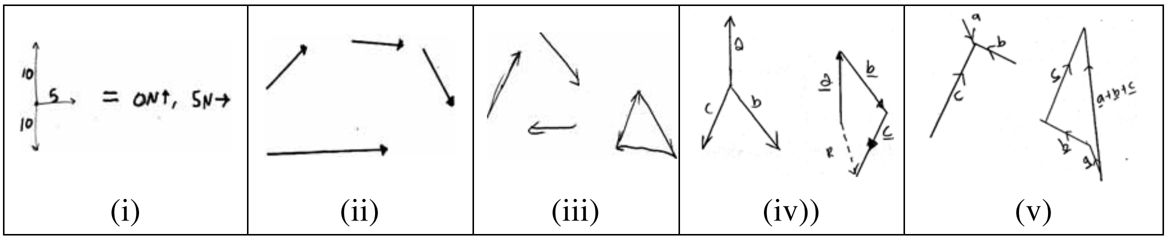


Fig. 6.29 Examples of responses to question set in the context of forces.

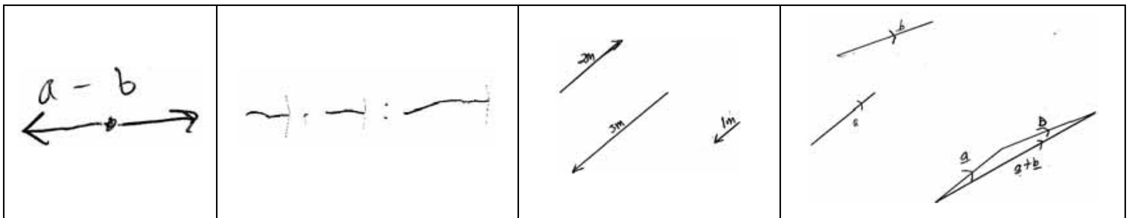
The response in part (i) was given stage 0 in the graphical mode and stage 2 in the symbolic mode. The student drew a very simple example and added components.

The response in part (ii) was given stage 2 in the graphical mode. The correct resultant is drawn separately without any indication of how it was obtained. Stage 0 was given for the symbolic mode (although student probably used the numerical addition to be able to draw the correct resultant, there is no indication of that in the test). Only the follow-up interview showed that the student added vectors numerically, by measuring the vertical and horizontal components, adding them together and drawing the answer graphically. This is a case, therefore, where what the student actually wrote in the test did not fully indicate his or her capacity. It is for this reason that the overall responses look for performance at the highest level shown by a student in all the questions, rather than average performances where individual cases may be given stage 0, merely because the student did not to use that mode explicitly.

Part (iii) was given stage 3 in the graphical mode and stage 0 in the symbolic mode. As the student drew his/her own vectors, with a very approximate drawing, it was possible that the resultant was 0. The follow up interview revealed that the student had the misconception that the three vectors should make a triangle.

Parts (iv) and (v) were both given stage 4 in the graphical mode. However part (iv) was given stage 0 in the symbolic mode, while part (v) was given stage 1 in that mode as the student added written letters. It was interesting to see that the student in figure 6.29 (v) worked in such a way that the response became 'singular' (with all three vectors directed into a single point).

The examples of responses to question 6 are shown in figure 6.30.



(i) (ii) (iii) (iv)

Fig. 6.30 Examples of responses to question set in the context of displacements.

The response in part (i) was given stage 0 in the graphical mode and stage 1 in the numerical mode as the student wrote the symbol **a–b**.

The response in part (ii) was given stage 1 in the graphical mode as the student added arrows in one dimension, and stage 0 for the symbolic mode.

The response in part (iii) was given 1 in the graphical response and stage 1 in the graphical mode as it seems that student just added signed numbers in one dimension.

The response (iv) was given stage 4 in the graphical mode and stage 1 in the symbolic mode.

6.4 Method of collecting qualitative data

Mason (1996) suggests that the sampling on the basis of chosen categories relevant to the research questions and one's theoretical position is called a *theoretical sampling*. My theory looked at students flexibility of thinking and the initial investigation suggests that students who present their work visually are often more flexible at this stage of their study and therefore I decided to choose students for the interviews on this basis.

During the interviews I considered different categories of questions described by Ainley (1988). It seems that the category described as testing questions (to find out if the subject knows the answer) and directing questions (provoking the subject to think further about a problem) are the most appropriate for this research.

The students were first asked how they attempted different questions and if they know any other way they could have answered (to find out if the subject knows the answer) and then some directing questions asking them where and how they used vectors in the past, if they know any rules for vector addition and what symbols they are familiar with (provoking the subject to think further about a problem).

As an additional qualitative data sample, the Mathematics and Physics teachers were interviewed about how they think student learn vectors and how vectors are taught in their subjects. This enabled further triangulation between what the students did and what their teachers expected them to do. The main study will also include interviews with teachers on how they think that the students will respond to give more information for triangulation purposes.

In addition, in the main study, the experimental lessons will be recorded to observe the students' development more closely.

6.5 Quantitative Analysis of the results

The students' results of the qualitative and quantitative analysis show a shift of the experimental class to being more 'graphical' than before and towards being more flexible. The vital delayed post-test missing from the pilot study was included in the main study. It was conducted half a year after the course to investigate long-term retention of ideas.

The analysis of the pilot study (published in part in Watson, Spyrou & Tall, 2002) indicates that a small number of students arrive in Year 12 to begin their A-level studies with an already-developed concept of free vector, and ability to apply it in all the above cases. However, the experimental lessons involving action on moving objects and reflective plenaries discussing free vectors moved many other students faster through the stages of the cognitive development of vector addition than the students who were not given this opportunity.

It is important to emphasise that the general principle of allocation of stages to students' responses had some effect on the analysis. Some cases are straightforward and the stages could be allocated straight from the theory designed in chapter 4. However, in ambiguous cases the students are given the highest category consistent with the response. As an example, some students, especially in the control group, lacked precision in their drawings to the extent that it was not obvious if they have the concept of addition or not. However they were given the benefit of the doubt and the highest stage consistent with the precise answer was awarded. The effect of this principle is that any bias in the recorded changes tends to benefit the control group rather than the experimental group, thus not falsely enhancing the effects of the experimental treatment.

The students' responses were divided into graphical responses and numerical responses as shown in chapter 4 in figures 4.6 and 4.7. Each response was given a level of development. The students' graphical and symbolic responses in the pre-test were plotted on the scatter graph as shown in figure 6.31.

The scatter-graph below (figure 6.31) shows the distribution of the stages given to students in both: experimental group A and control group B in the pre-test.

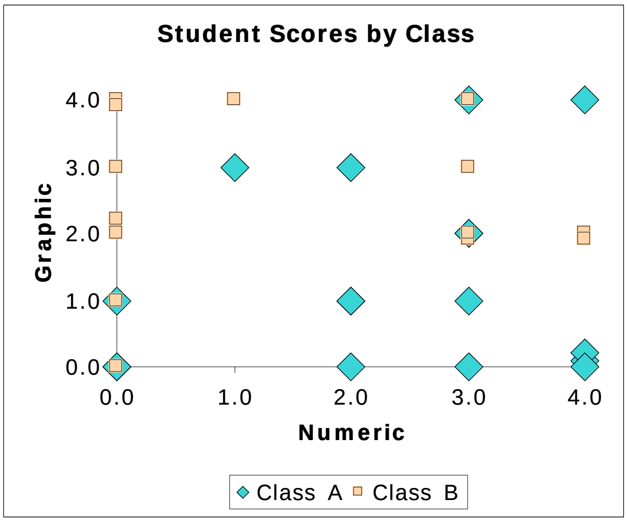


Fig. 6.31 Results of pilot pre-test.

The experimental group A is marked with rhombuses and the control group B is marked with squares. If we look at the 'squares' of group B, it can be noted that most of them are close to the vertical axis representing graphical responses. On the other hand more of the 'rhombuses' of group A seem to be closer to the horizontal axis, indicating numerical/symbolic responses. This difference was confirmed by statistical analysis. By comparison of the means and standard deviations, group B is more graphically biased than group A in the pre-test. The means and standard deviations of the two groups' graphical scores are μ_A=0.7, s_A=0.6 and μ_B=1.5, s_B=1.1. Using the t-distribution, there is statistical evidence ($t=2.84$, $p<0.05$) to suggest that group B is more graphically biased.

It was hypothesised that, through providing students in group A with the embodied experience translated into symbolism, we could move more of them into right top corner of the scatter graph. Both experimental and control treatments involved a substantial experience of graphical representations and addition of forces as vector quantities, so both groups would be expected to change in this direction. A t-test conducted on the improvement of responses in the graphic mode shows that the changes were as follows:

Group A: $t=5.9$ significant at $p<0.0005$;

Group B: $t=2.4$ significant at $p<0.025$.

Both groups therefore made statistically significant improvements, but the changes in group A were greater than those in group B. In part, this may be attributed to the better final results of group A, but there is also a contribution to the difference which occurs because group B was already more graphically orientated in the pre-test.

When looking at the stages 3 and 4 of the cognitive development, both groups started with about 65% of students in the pre-test responding at those stages in one or both modes of operation. However in the post test 95% of student in the experimental group responded at stage 3 or 4 in either mode and at the same time 72% of students responded at stage 3 or 4. This gives χ^2 =3.16, which is significant at $p\leq 0.1$.We can see in the scatter-graph in figure 6.32 that only one student in group A responded at the stage lower than 3 in the graphical mode while 5 students in group B answered at the stage below 3 in the graphical mode.

The changes in symbolic responses are not significant. However, there was a greater difficulty in assigning stages as what the students write does not always represent what they are capable of doing. There is a great need therefore for in-dept interviews to study this aspect. This will play a major role in the main study.

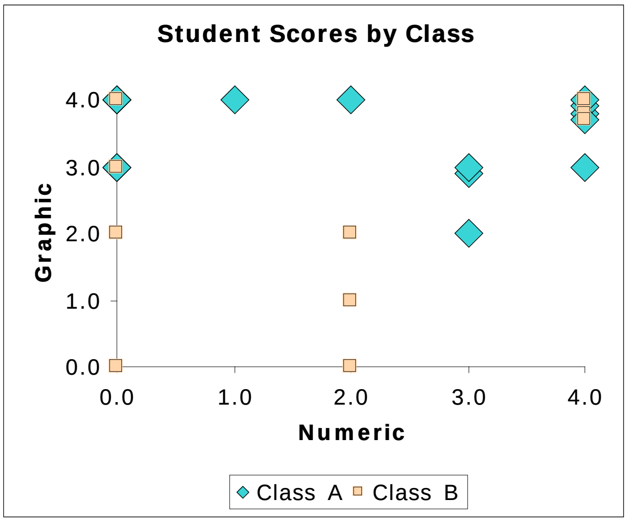


Fig. 6.32 Results of pilot post-test

The scatter graph in figure 6.32 shows that in group A, most students moved up to the third or fourth stage of the cognitive development in the graphical mode and more students moved to the top-right corner. Meanwhile, the results of group B split into two main groups, one of which occupies the top right corner, with others who continue to cling to the vertical graphic axis with zero numeric score.

From the initial interviews there was an indication that students who are more 'graphical' are also more flexible and think more conceptually. For instance, they use graphical methods in a more efficient way in the questions given in the test, while students who are more 'numeric' or symbolic, tend to use the symbolic method procedurally and without flexibility. This would suggest that before the experiment starts, the Experimental Group A are more numeric and probably less flexible than the Control Group B.

6.6 Conclusions

The tests proved adequate information about students' development and is going to be used in the main study. It also showed that the change in the experimental group was significantly different to the change in the control group. This difference shows straight after the course has been completed. The main study hopes to prove long-term concept stability in the experimental group and therefore the difference should become greater in the delayed post-test.

The method of collecting data showed that all of the responses could be classified although the classification of stage 0 is not always completely clear. However it was felt that five stages including the zero stage gave a fair overall indication of the student development. What matters in this study is the movement through the cognitive stages to the higher levels, and this is the main focus of attention rather than a deeper study of the pre-conceptual development.

It was difficult at times to give students a higher stage based on the response from the test, knowing that the preliminary interviews show the possibility of a lower stage. The moderation of the stages caused also some difficulty as it was a tendency to assume what student might have wanted to say instead of keeping strictly to the work shown. When the consensus could not be achieved between two teachers giving stages the higher stage was adopted. In order to maintain a consistency in assigning stages, the same method will be used in the main study.

It was also decided that although the post-test questionnaire is going to be applied in the classes following the experiment, the main focus of attention will be on the delayed post-test which will be given to students after their holidays and 11 months after the experimental lessons (at the beginning of year 13). This should test if the changes caused by the experimental lessons can be sustained for a longer period of time.