Chapter 7
Main Study: Quantitative Data Analysis

7.1 Introduction

The emphasis of the experimental lessons was directed to compressing the embodied actions into process by focusing on the notion of *effect* (if two actions have the same effect then they are considered as giving the same process).

Reflective plenaries were introduced for the Experimental Group in the Teaching Experiment to concentrate on the effect of different procedures.

The intention was to test the following hypothesis:

> **Main Hypothesis**: Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

The intention of the teaching was to help the students appreciate the equivalence of '**free vectors'** with **the same magnitude and direction** and the flexible use of equivalent vectors for vector addition. The testing of this hypothesis was performed by designing a questionnaire as a tool to test the changes in the stages of cognitive development (as shown in chapter 3, figures 3.16 and 3.17) and discussed in detail in chapter 6. The main hypothesis infers:

> **Hypothesis 1**: Students, who were involved in experimental lessons, are expected to rise through the cognitive stages further than students who are not exposed to the experimental lessons.
>
> **Hypothesis 2**: Students who were helped in building a concept of a free vector are expected to be more able to:
>
> **(a)** add vectors in singular cases, not just generic ones;
> **(b)** use free vectors independent of the context;

(c) realise that the commutative law applies to vector addition.

Hypothesis 3: Students who can concentrate on the *effect* of actions rather than actions themselves are more likely to build the concept of free vector as a cognitive unit, which can be used by students after a longer period of time and not only just after the experiment.

For this reason, the main comparison of the data will be done between the pre-test and the delayed post-test.

This chapter tests the main hypothesis through the outcome of the analysis conducted at three different stages of the research according to the Methods and Methodology developed through chapters 4, 5 and 6. This chapter will present the quantitative analysis of the data from the questionnaire, which will then triangulated with the qualitative data from both teachers and students (chapters 8, 9).

Hypothesis 1 will be tested using the data related to the students overall performance on the questionnaire with respect to the concept of vector and vector addition. Hypothesis 2 will be tested using questions specifically focused on (a) singular questions, (b) questions in different contexts (c) questions that may be solved using the commutative law. Hypothesis 3 will be tested quantitatively by focusing on the same data over the period from pre-test to immediate post-test through to delayed post-test. It will later be tested qualitatively by analysing the responses of students in the interviews reported in chapter 9.

7.2 Quantitative Data Analysis of Understanding the Concept of Vector and Vector Addition

The quantitative analysis arises from the data collected in 3 tests conducted before the course (pre-test, T1), straight after the course (post-test, T2) and half a year after the course (delayed post-test, T3) with two groups of students: Group A (experimental) and Group B (control). Both groups had 17 students each.

The tests, given at different times of the year, are considered to be indicators of the students' cognitive development stage. Therefore the change of that stage from one test to the next is considered as an indication of the students' cognitive development.

The general/overall distribution of students between different stages of cognitive development in understanding of vector and vector addition can be viewed in parts 7.1.1 and 7.1.2 of this chapter. This analysis considers students' understanding of vector and vector addition at three stages of their development into studying Mechanics, without looking at how they can apply their knowledge in questions involving singular cases or different contexts cases. The later parts of the chapter show the distinctions between students' responses to singular questions (section 7.1.3), to questions set in two different physical contexts (displacement and forces) (section 7.1.4), and to questions that may use commutative law of vector addition in the solution process. The data was built from the students' responses using the methods discussed in detail in chapter 6.

7.2.1. The General Case: Understanding the Symbol of Vector

Tables 7.1 and 7.2 show the number of students at different stages of the cognitive development ladder captured in the three tests. Table 7.1 shows the categorisation of graphical responses and 7.2 the categorisation of symbolic responses. There were 17 questions and sub-questions in total and the student had to achieve their highest stage twice to be given it (as described in detail in chapter 6.2).

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	3	17	17	6	10	13
3	9	0	0	6	4	3
2	5	0	0	1	3	1
1	0	0	0	1	0	0
0	0	0	0	3	0	0
TOTAL	17	17	17	17	17	17

Table 7.1 Graphical responses to test questions

From the table 7.1 it would appear that there were more students responding at stage 4 in group B than in group A at the beginning of the year 12 (T1), however, there were also 3 students in group B responding only at stage 0. At the same time, all students in group A reached stage 4 in the post test (T2) and retained their knowledge until the delayed post-test (T3). Meanwhile, in Group B, only 10 out of 17 students reached stage 4 in the post-test (T2) and that number increased to 13 in the delayed post-test (T3).

The two-tail t-test performed for each group on students' changes in the stage of the graphical cognitive development between test 1 and test 3 shows:

$t=5.37$ which is highly significant ($p<0.01$) for group A and

$t=3.83$ which is significant ($p<0.01$) for group B.

This indicates that both groups have improved their responses of the graphical representation of the vector.

The next table (table 7.2) shows the results on the basis of the symbolic representation responses.

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	6	7	6	4	4	4
3	1	3	1	2	5	1
2	4	0	2	5	2	7
1	5	5	4	5	4	2
0	1	2	4	1	3	3
TOTAL	17	17	17	17	17	17

Table 7.2 Symbolic responses to test questions.

From table 7.2 we can observe that the changes between the pre-test and delayed post-tests are not substantial and the t-test performed on student's changes in the stage of symbolic cognitive development proved not significant.

If we look at the scatter graphs in figures 7.1-7.3, we can confirm that there are no significant differences between the experimental group A and the control group B at the three stages and that both groups developed their understanding of the symbol of vector between the pre-test and the delayed post-test.

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4		BB	A BBB	A	A B
stage 3	A	AAA B	A B	BB	AAA BB
stage 2		AA	AA		A B
stage 1		B			
stage 0	B	B	B		A

Fig. 7.1 Scatter graph of responses to all pre-test questions (vector)

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	A BB	AAAAA B	AA BBBBB		AAAAAAAA BBBB
stage 3		B	B	A	
stage 2	BB	B			
stage 1					
stage 0					

Fig. 7.2 Scatter graph of responses to all post-test questions (vector)

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	AAAA B	AAAA B	AA BBBBBB	A B	AAAAAA BBBB
stage 3	B	B	B		
stage 2	B				
stage 1					
stage 0					

Fig. 7.3 Scatter graph of responses to all delayed post-test questions (vector)

Using the classification given in table 6.1 (chapter 6), figure 7.1 shows that in the pre-test T1 there were 2 students from group A categorised as ***uni-modal*** and 4 classified as graphically-orientated ***higher uni-modal***; 5 students were classified as ***multi-skilled***; 5 students were already in the ***versatile*** or ***fully-integrated*** categories. At the same time, in group B, one student was in the intuitive category; 3 in the ***uni-modal*** category; 3 in the graphically orientated ***higher uni-modal***; 5 students were in the ***multi-skilled*** category and 5 were already ***versatile*** or ***fully-integrated.*** Both groups were therefore similar in their cognitive development of vector and flexibility in using the graphical or symbolic modes of operation.

The χ^2-test could not be used as the expected numbers are too low. However, it can be seen that their general development of understanding the symbolic and graphical representation of vector remained similar even at the time of the delayed post-test, with the graphical categories all high and the numerical categories spread out over the full range.

7.2.2. The General Case: Understanding Vector Addition

Both sets of students had considerable experience with the concept of vector in Mechanics in dealing with forces. It is therefore not surprising that they have improved in both groups in understanding of the symbol of vector. However, how well they understand the symbol in order to be able to manipulate it was tested through the questions asking them to add vectors. The analysis of addition of vectors is considered in this section.

Tables 7.3 and 7.4 below show the numbers of students responding at different stages of the cognitive development of the concept of vector addition in a graphical and symbolic mode of representation.

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	3	16	2	4	9
3	2	13	1	9	9	4
2	10	1	0	1	1	4
1	0	0	0	2	3	0
0	5	0	0	3	0	0
TOTAL	17	17	17	17	17	17

Table 7.3 Graphical responses to the test questions

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	3	4	0	0	1
3	6	3	1	6	2	1
2	2	2	0	3	5	4
1	6	5	5	7	3	6
0	3	4	7	1	7	5
TOTAL	17	17	17	17	17	17

Table 7.4 Symbolic responses to the test questions

The data from the pre-test T1 shows that, in the graphical mode, only 2 out of 17 students in group A responded in the two highest stages (3 and 4), compared to 11 out of 17 students in group B. However, in the symbolic mode, the two groups had very similar distributions, each with 6 students at stage 3 and none at stage 4.

In the delayed post-test T3, in the graphical mode, the number of students in group A responding at stages 3 and 4 increased from 2 to 17, while in group B the numbers of students stayed nearly the same (a small increase from 11 to 13).

The significance of the changes can be determined using the two-tail t-test. The t-test taken for the graphical changes between the test T1 and T3 shows:

$t=3.83$ which is significant ($p<0.01$) for group A and

$t=0.348$ which is not significant for group B.

The two-tail t-test conducted on the symbolic responses show that the changes between pre-test and delayed post-test in not significant for either group.

Between the time of the pre-test and the post-test, both groups did a lot of work in their Mechanics lessons on addition of forces presented in a graphical way and it should be noted that at the stage of the post-test, both groups seem to be at a similar stage of their cognitive development of vector addition. However it is very noticeable that students in group A achieved long-term concept stability: 16 out of 17 students responded at stage 4 in the graphical mode in test T3. Group B also improved, and 9 out of 17 reached stage 4.

The scatter graphs in figures 7.4-7.6 show the students' development through the second stage of categorisation: from ***intuitive*** and ***uni-modal*** to ***higher uni-modal***, ***multi-skilled***, ***versatile*** and ***fully-integrated.***

graphical mode		symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
	stage 4		**BB**			
	stage 3	**A**	**BB**	**BBBB**	**A BB**	**B**
	stage 2	**AAA**	**AAAAA B**	**A**	**A**	
	stage 1		**B**		**B**	
	stage 0	**AAA B**	**BB**	**A**	**A**	

Fig. 7.4 Scatter graph of responses to all pre-test questions on addition

graphical mode		symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
	stage 4	**AA BB**	**A**	**BB**		
	stage 3	**A BBB**	**AAAA B**	**AA BBB**	**AAA BB**	**AAA**
	stage 2	**A B**				
	stage 1	**B**	**BB**			
	stage 0					

Fig. 7.5 Scatter graph of responses to all post-test questions on addition

graphical mode		symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
	stage 4	**AAAAAAA B**	**AAAA BBBB**	**BBBB**	**A**	**AAAA**
	stage 3	**BB**	**A**		**B**	**B**
	stage 2	**BB**	**BB**			
	stage 1					
	stage 0					

Fig. 7.6 Scatter graph of responses to all delayed post-test questions on addition

In the pre-test T1, in group B, four students began in the higher uni-modal (graphical mode) category, four in multi-skilled (graphical mode) and 3 in the versatile category, while, at the same time, in group A only two students were in any of those categories.

However by the time of the delayed post-test T3, the picture has changed substantially as far as group A is concern. In group B there are still four students in the uni-modal category while in group A, 16 out of 17 students responded at stage 4 (mainly graphically) and 4 of those are in the fully-integrated category.

These results are the evidence for hypotheses 1 and 3. The students who were involved in the experimental lessons rose through the cognitive stages further than students who were not exposed to the experimental lesson and their conceptual understanding worked after a longer period of time and not just after the experiment.

7.2.3. Singular Cases: Understanding Vector Addition

Hypothesis 2(a) states that the difference should show when looking at students' flexibility tackling singular questions. As we already know from section 7.1.1, there may be little difference in the overall spectrum of understanding of the concept of vector in the two groups, so we only analyse whether the singular questions cause a difference to the ways in which students carry out vector addition.

Tables 7.5 and 7.6 give a summary of students' responses to the singular cases.

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	1	12	2	0	7
3	1	9	4	1	10	3
2	4	6	1	1	3	2
1	4	1	0	4	1	0
0	8	0	0	9	3	5
TOTAL	17	17	17	17	17	17

Table 7.5 Graphical responses to the singular questions

The t-tests performed on students' changes in the stage of the graphical cognitive development between the pre-test and the delayed post-test show:

t=3.13 which is significant ($p<0.01$) for group A and

t=1.3 which is not significant for group B.

This supports hypothesis 2(a) that there is a statistically significant improvement in group A but not in group B.

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	0	4	0	0	1
3	0	4	1	2	2	1
2	2	4	0	2	2	2
1	5	3	4	4	3	7
0	10	6	8	9	10	6
TOTAL	17	17	17	17	17	17

Table 7.6 Symbolic responses to the singular questions.

The t-tests performed on students' changes in their stage of symbolic cognitive development between pre-test and delayed post-test shows that the changes are not significant for either group.

The data in these tables reveals that prior to the experimental study, in the pre-test T1, only one student in group A was able to respond to singular questions at stages 3 or 4 in the graphical mode and no student replied in these stages in the symbolic mode. At the same time, in group B, three students were able to respond in graphical mode and two in symbolic mode at stages 3 and 4.

This changed substantially in the graphical mode by the time of the delayed post-test. In group A, 16 out of 17 students responded at stages 3 and 4 (of whom 12 were at stage 4), while in group B, 10 students were able to respond at those stages (with 7 at stage 4).

It must be emphasized that the post-test was carried out straight after the mechanics and physics courses dealt with forces in vector forms and after the students in group A had their experimental lessons, while the delayed post-test was carried out half a year after that time. The immediate post-test does not show any significant differences between the groups, however significant changes occur later, in the delayed post-test, which indicates a long-term stability of conceptual growth in group A.

The scatter graphs in figures 7.7-7.9 show the students' development through the second stage of categorisation: from ***intuitive*** and ***uni-modal*** to ***higher uni-modal***, ***multi-skilled***, ***versatile*** and ***fully-integrated.***

graphical mode						
	stage 4		**BB**			
	stage 3	**A**	**BB**	**BBBB**	**A** **BB**	**B**
	stage 2	**AAA**	**AAAAA** **B**	**A**	**A**	
	stage 1		**B**		**B**	
	stage 0	**AAA** **B**	**BB**	**A**	**A**	
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.7 Scatter graphs of responses to singular pre-test questions

graphical mode						
	stage 4			A		
	stage 3	AAAA BBBBBB	AA BB	AA BB	A	
	stage 2	AA B	A	A	AA BB	
	stage 1	B			A	
	stage 0	BB	B			
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.8 Scatter graphs of responses to singular post-test questions

graphical mode						
	stage 4	AAAAAA BB	AA BBB	B	A	AAA B
	stage 3	A B	AA B	B		A
	stage 2	A	BB			
	stage 1					
	stage 0	BBB	B		B	
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.9 Scatter graphs of responses to singular pre-test questions

The testing of the difference between the proportions of the students in the *intuitive/ uni-skilled* area shows significant difference in favour of group A (χ^2 = 2.97 significant at p<0.01). In the pre-test 12 students out of 17 from group A but only 5 out of 17 student from group B were in the *intuitive/uni-skilled* area. However in the delayed post-test only 1 student in group A was in this area compared with 6 students in group B. This supports hypotheses 2(a) and 3, in that group A students' conceptual knowledge of vector addition was more firm by the time of the delayed post-test and they could apply it more flexibly, even in the singular cases. On the other hand, in comparison with group A, a greater number of students in group B had a limited procedural view of vector addition as they could only answer generic questions and had problems with singular examples.

7.2.4. Different contexts: Understanding Vector Addition

This section considers the students' responses to the questions set in two different contexts. The intention is to check whether there is a significant difference between the improvement in marks of the experimental and control groups in their solution of problems in different contexts.

Hypothesis 2(b) states that the difference should show when looking at students' flexibility in tackling different contexts. The analyses in this part of the chapter show what happens in the case of responses to questions set in two different contexts (forces and displacements). Tables 7.7 and 7.8 present students' responses to the questions in the test set in two different contexts.

Graphical cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	0	8	0	0	2
3	0	9	3	2	3	5
2	1	2	2	0	3	3
1	1	5	4	0	2	3
0	15	1	0	15	9	4
TOTAL	17	17	17	17	17	17

Table 7.7 Graphical responses to questions set in different contexts

The t-tests performed on students' changes in the stage of the graphical cognitive development between the pre-test and the delayed post-test show:

t=8.71, which is highly significant ($p<0.01$) for group A and

t=2.17, which is significant ($p<0.05$) for group B.

This supports hypothesis 2(b) that Group A made a more significant overall improvement in their stages of cognitive development than Group B.

Symbolic cognitive stage	Group A			Group B		
	T1	T2	T3	T1	T2	T3
4	0	0	0	0	0	0
3	0	1	1	2	1	0
2	2	1	0	6	4	4
1	2	1	1	2	4	8
0	13	14	15	7	8	5
TOTAL	17	17	17	17	17	17

Table 7.8 Symbolic responses to questions set in different contexts

The t-test performed on students' changes in the stage of the symbolic cognitive development between the pre-test and the delayed post-test was insignificant for both groups.

From the data in Table 7.7 it can be seen that in the delayed post-test, 11 out of 17 students in group A answered the questions at stage 3 or 4 of the graphical mode. Taking the two stages together, the number of students has not changed. However, if we just look at the stage 4, the numbers changes from 0 in the pre-test to 8 in the delayed post-test. At the same time in the delayed post-test, 7 students out of 17 in group B managed to answer the questions at the stage 3 or 4, but only two students responded at stage 4.

Table 7.8 shows that group A students were less inclined to respond symbolically in all three tests than group B. In addition, the scatter graphs below (figures 7.10-7.12) show that, in pre-test T1, the students in both groups did not show any signs of flexibility. In the delayed post-test T3, the students' answers are at higher cognitive stages than in the previous tests and their answers are mainly in the graphical higher uni-modal category. A t-test showed no significance in either of mode of operation. To make a more subtle analysis, it was decided in the second type of categorisation to look at the two different contexts separately.

Both groups worked on the topic of forces for the same number of lessons and covered the same questions from the textbook and therefore the results should be similar at all stages if the experimental lessons had no consequence on group A students' cognitive development. The three figures below (figure 7.10 – 7.12) show the results of students responses to the question set in the context of forces.

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4					
stage 3	**BBB**				
stage 2	**AA** **BB**		**B**	**BB**	
stage 1	**A**			**BB**	
stage 0	**AAAAAAAAAAA** **BBBB**	**A** **B**	**A** **B**	**AA** **B**	

Fig. 7.10 Scatter graph of responses in the context of forces, pre-test

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	**AAAAAA** **BB**	**AAA** **B**			**A**
stage 3	**A** **B**	**A**			**B**
stage 2	**A** **BB**	**B**			
stage 1	**A**	**A**		**A**	
stage 0	**AA** **BBB**		**BBBBB**		**B**

Fig. 7.11 Scatter graph of responses in the context of forces, post-test

graphical mode	symbolic mode: stage 0	stage 1	stage 2	stage 3	stage 4
stage 4	**AAAAAA** **BBBB**	**AAAAA** **B**			**A**
stage 3	**AAA** **BB**				
stage 2	**A**				
stage 1	**A** **BB**		**BBBB**		
stage 0			**BBB**		**B**

Fig. 7.12 Scatter graph of responses in the context of forces, delayed post-test

The students operating at the combined lower stages (0-2) of the cognitive development in the graphical and symbolic modes fall into the *intuitive/uni-modal* area of the chart. The students operating at higher stages of the cognitive development (3-4), in either of the modes, and those operating at stage 2 in both modes, fall into the *higher uni-modal, multi-skilled, versatile and fully integrated* area of the chart. The χ^2-test compared the differences between both groups in each area:

> pre-test results (χ^2 = 5.25 significant at $p<0.05$) showed a significant difference between the two groups **in favour of group B** being in the higher area of the graph;
>
> post-test results (χ^2 = 2.95 not significant at $p<0.05$) shows that there was no significant difference between the groups;
>
> delayed post-test results (χ^2 = 4.84 significant at $p<0.05$) shows an even greater difference between the two groups **in favour of group A** being in the higher area of the graph.

These results indicate that group A, in comparison with group B, gained conceptually from the experimental lessons in the context of vector as force, and sustained their knowledge between the post-test and the delayed post-test. The difference between the groups changed from Group B being significantly higher in the pre-test to Group A being significantly higher in the delayed post-test. It is relevant that there was no significant difference between the groups in the immediate post-test. The gain is long term rather than short-term.

The first meeting of the concept of vector in the Mathematics Syllabus happens in the context of translation — displacement in physical terms. The students in both groups should have therefore had a similar competence at the beginning of year 12. The experimental lessons (which focused on translations) should have a positive effect on group A students in their cognitive development of vector addition and therefore the difference between the groups should be significant in the post-test and the delayed post-test.

The three figures below (figures 7.13-7.15) show the results of students responses to the question set in the context of displacement.

graphical mode	stage 4		BB			
	stage 3					
	stage 2		A			
	stage 1	A	A	BB		
	stage 0	AAAAAAAAAAAAA BBBBBBB	A BB	BBBB		
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.13 Scatter graph of responses in the context of displacements, pre-test

graphical mode	stage 4	AAAAAA B	A B			A B
	stage 3					
	stage 2	B				
	stage 1	A B	AA BB	B		A
	stage 0	AAA BBBBB	AA B	BB		B
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.14 Scatter graph of responses in the context of displacements, post-test

graphical mode	stage 4	AAAAA BBB	AAA			A
	stage 3	AAA B				
	stage 2	A BB		B		
	stage 1	AA B	B			A
	stage 0	A BBBBB	B	BB		
		stage 0	stage 1	stage 2	stage 3	stage 4
		symbolic mode				

Fig. 7.15 Scatter graphs of response in the context of displacements, delayed post-test

It can be seen from the scatter graphs above that in the post-test there was a tendency for both groups to respond symbolically to the question on displacement. Most of the responses are clustered around the symbolic axis but at the low stages. In the post-test this changed and the students moved more towards graphical responses. By the delayed post-test the tendency to give only graphical responses increased even further and most students are clustered around the graphical axis. Only one student (from group A) answered in the fully integrated category and only one responded at the highest symbolic level, with a low graphic score.

The χ^2-test on the difference between students in two groups gave the following results:

> pre-test results (χ^2 = 0.53, not significant), no significant difference between the two groups;
>
> post-test results (χ^2 = 1.99 not significant), no significant difference between the two groups;
>
> delayed post-test (χ^2 = 5.78, significant at $p<0.05$) shows a significant difference between the two groups in favour of group A.

These results support Hypothesis 2(b), showing a significant improvement long-term in favour of group A, with 13 out of 17 students benefiting from the experimental lessons so that they could use a vector as a mathematical mental concept to solve problems in different contexts. The results also show a long-term improvement which supports hypothesis 3.

7.2.5 The commutative law in vector addition

Hypothesis 3 also states that the students who were helped in building a concept of a free vector can realise that the commutative law applies to the vector addition.

The students had the opportunity to use commutative law of vector addition in four questions in the test. The numbers of students in both groups using this opportunity in three tests can be seen in table 7.9 below:

	Group A	Group B
Pre-test	0	4
Post-test	7	6
Delayed Post-test	12	5

Table 7.9 Responses using the commutative law of addition

There is little change in the results of group B. However, in the pre-test, group A students did not use the commutative law at all, but by the time of the delayed test, 12 out of 17 students used it, which is 70% of students in comparison with 29% in group B. As the understanding of the commutative law is related to the use of vectors as free vectors, this is consistent with the interpretation that 70% of students in group A have a concept of free vectors, compared to only 29% of students in group B.

The χ^2-test run on the difference between students in two groups gave the following results:

pre-test results ($\chi^2 = 4.53$, significant at $p<0.05$) shows a significant difference between the two groups in favour of group B;

post-test results ($\chi^2 = 0.5$, not significant) shows no significant difference between the two groups

delayed post-test ($\chi^2 = 5.76$, highly significant at $p<0.05$) shows a significant difference between the two groups in favour of group A.

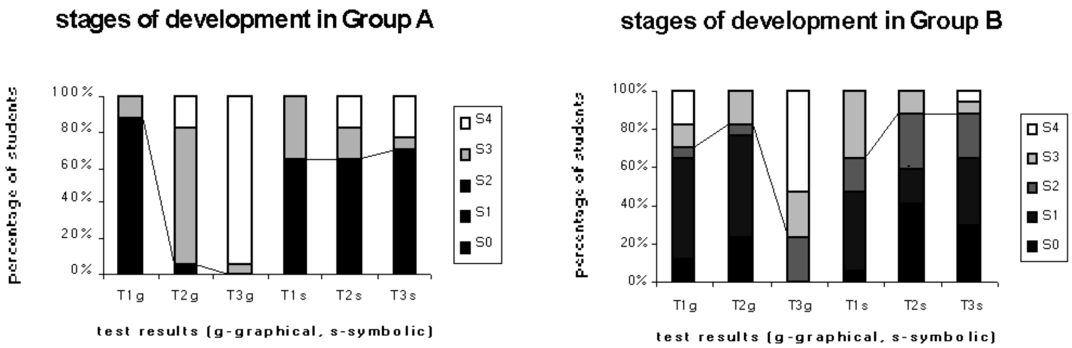
Hypothesis 2(c) is therefore also confirmed.

Therefore hypotheses 2(a), 2(b) and 2(c) are all supported, with significant improvements by group A over group B in handling singular cases, questions in different contexts, and the use of the commutative law. This analysis also gives quantitative support for Hypothesis 3, in that all three cases showed a significant long-term improvement in the performance of group A on the delayed post-test. It would seem that the embodied experiences may have given deep cognitive support that allowed the concept to continue maturing over a long period of time.

7.3 Summary of the results

The results so far show a significant improvement in the performances of group A in the graphical mode, but little significant improvement in the numerical performance. In this section we review the data from each test in turn to see if the evidence reveals any further evidence of differences between the two groups.

The results of the study, revealing differences between the experimental and control groups can be represented by the graphs in figures 7.16 and 7.17. The two graphs in figure 9.7 show the students' cognitive development through 4 stages in vector addition at three successive points in the year (T1, T2 and T3), with Group A on the left and Group B on the right. T1g, T2g and T3g represent the graphical results in the three tests, and T1s, T2s and T3s represent the symbolic results.



(a) (b)

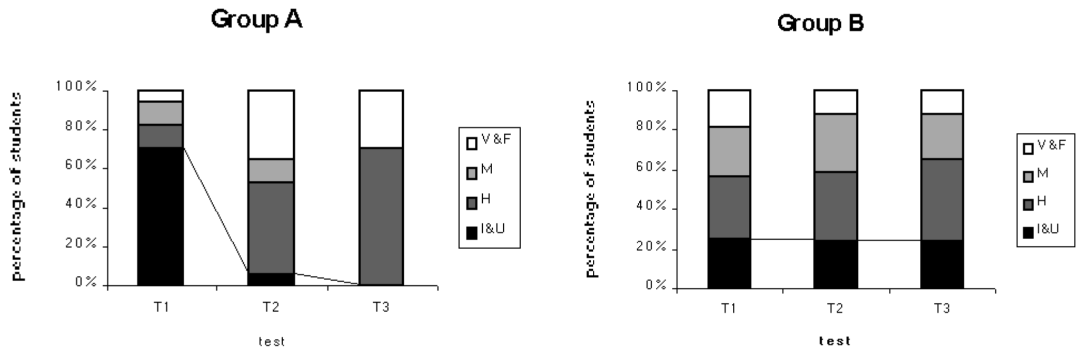
Fig. 7.16 Comparative General Developments of Groups A and B

The stages in each column are represented vertically in successive shades from stage 0 (black) at the bottom to stage 4 (white) at the top. The lines joining successive columns show the changing levels of the point between the two higher stages (3 and 4) and the three lower stages (0, 1, 2).

The percentages scoring in the two higher stages of performance (stage 3 and 4) in the graphical development of Group A increase from around 12% in the first test to 94% in the third test. The corresponding graphical results in Group B increased from around 65% up to 76%. Thus, in graphical development, Group A started *below* Group B, yet ended up *above* them.

In the numerical development, group A started with about 35% of students in the higher stages and finish at about 29%. At the same time group B also started at about 35% of students at the higher stages of development and finish with about 12%. In this case the changes are not statistically different, but there is a tendency for high-level numeric responses to decrease during the teaching.

Figure 7.17 combines the graphic and numeric information based on the categories developed in figure 3.1 of chapter 3 to show the percentages of students belonging to each category (I: *intuitive,* U: *uni-modal*, H: *higher uni-modal*, M: *multi-skilled*, V: *versatile* and F: *fully integrated*). In the figure the *intuitive* and *uni-modal* categories are integrated into a single category, as are the *versatile* and *fully integrated* categories.



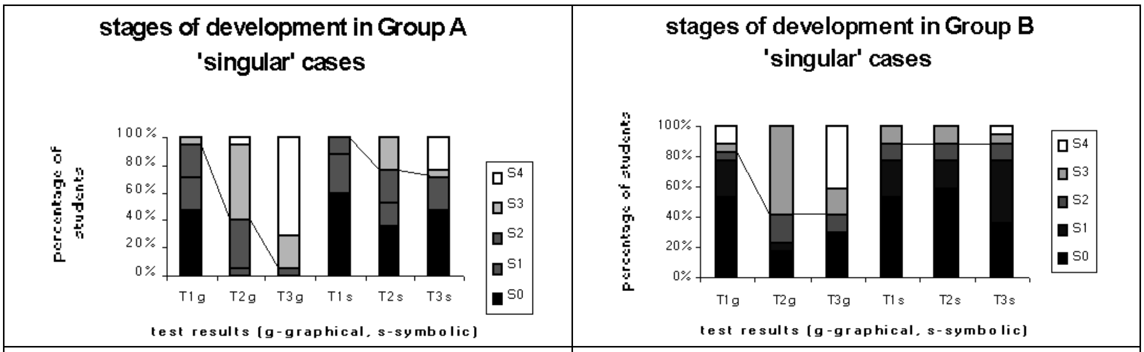
(a) Experimental group A (b) Control Group B

Fig. 7.17 Development of students through combined categories

In the pre-test (T1), group A appeared to be more intuitive/uni-modal (~70%) in comparison with group B (~23%), and therefore at a lower stage of cognitive development. However, by the time of the delayed post-test (T3), no students in group A remained intuitive/uni-modal. At the same time in group B, at the time of the pre-test, a lower percentage of students were intuitive/uni-modal (~23%), however, this number did not change throughout the year.

These results confirm the hypotheses stated at the opening of the chapter. The main hypothesis that the experimental treatment focusing on ‘effect’ would be more likely to lead to the notion of a free vector used in a versatile manner is supported by statistical data that Group A rise further through the cognitive stages (hypothesis 1) and that these gains are retained over the longer term (hypothesis 3).

The differences between both groups in case of the singular questions can be seen in figures 7.18 and 7.19.



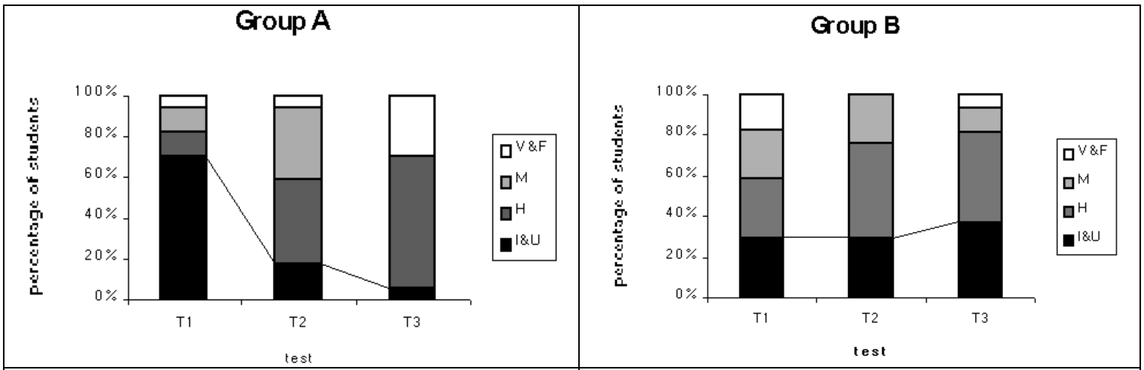
(a) (b)

Fig.7.18 Comparative development of Groups A and B (singular cases)

The percentages scoring in the two higher stages of performance (stage 3 and 4) in the graphical development of Group A increased from about 6% in the first test to about 94% in the third test. The corresponding graphical results in Group B increased from about 18% to about 59%. Thus, in graphical development, Group A again started *below* Group B, yet ended *above* them.

In the symbolic representation, Group A increased their performance (at stages 3 and 4) from about 6% to about 30%, while at the same time Group B stayed consistently at about 12%.

The differences between the two groups can also be highlighted when the responses are combined into another set of categories, presented in figure 7.19.



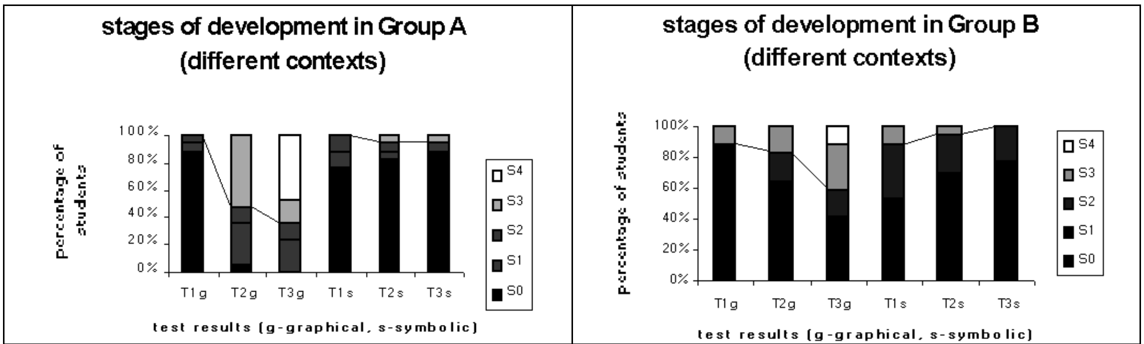
(a) (b)

Fig. 7.19 Development of students through combined categories (singular cases)

In the pre-test (T1) Group A appeared to be more intuitive/uni-modal (~70%) in comparison with group B (~30%), and therefore Group A was at a lower stage of cognitive development. However at the time of the delayed post-test (T3), in Group A, only small number (~6%) of students, remained intuitive/uni-modal. At the same time in Group B the number of the intuitive/uni-modal students increased slightly (~35%).

These results confirm further the hypotheses stated at the opening of the chapter. The main hypothesis that the experimental treatment focusing on 'effect' would be more likely to lead to the notion of a free vector used in a versatile manner, not just in generic cases but also in singular cases is supported by statistical data that Group A rise further through the cognitive stages (hypothesis 1, and hypothesis 2(c)) and that these gains are retained over the longer term (hypothesis 3).

There are also differences in case of the questions set in different contexts, which can be observed in figures 7.20 and 7.21.



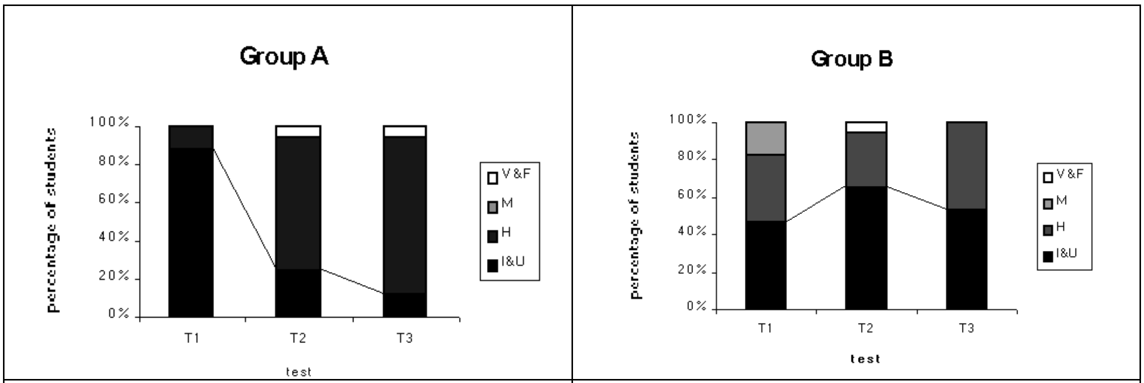
(a) (b)

Fig.7.20 Comparative developments of Groups A and B (different contexts)

The percentages scoring in the two higher stages of performance (stage 3 and 4) in the graphical development of Group A increased from about 0% in the first test to about 65% in the third test. The corresponding graphical results in Group B increased from about 12% to about 41%. Thus, in graphical development, Group A started *below* Group B, and again ended up *above* them.

In the symbolic representations, there very insignificant changes in both groups (figure 7.20 (b)).

The differences between the two groups can also be noted when the responses were combined into another set of categories, presented in figure 7.21. The changes are only shown in case of context of forces as both groups did an equal amount of work in that context.



(a) (b)

Fig. 7.21 Development of students through combined categories (context of forces)

In the pre-test (T1) Group A appeared to be more intuitive/uni-modal (~88%) in comparison with group B (~47%), and therefore Group A was at a lower stage of cognitive development. However at the time of the delayed post-test (T3), in Group A, only small number (~11%) of students, remained intuitive/uni-modal while in Group B the number of the intuitive/uni-modal students increased slightly (~53%).

7.4 Summary

The quantitative data analysis reveals statistical support for the hypotheses stated in the opening of the chapter. The improvements occurred mainly in the graphical mode, with no statistically significant changes in the numerical mode. In particular, the students in group A showed little evidence of moving to the fully integrated area and responded mainly in the graphical mode. The reason for this may be that, since the questions were easier to answer in the graphical mode, and experimental students gained confidence in operating in this mode, they chose this means of response. This reason cannot be confirmed by the written evidence alone, but it will be tested in interviews with a sample of students (discussed later in chapter 9).

The difference between the two groups is apparent in the responses to the questions involving singular examples (hypothesis 2(a)) and problems set in different contexts (hypothesis 2(b)). The evidence of the performance of the group as a whole is consistent with the hypothesis that the experimental group students are more flexible in adapting their knowledge to different circumstances even after a longer period of time. The evidence is also consistent with the hypothesis that students in group A will construct the notion of free vector to a greater extent than group B, as they show greater ability in applying the concept of the commutative law to vector addition.

In my experience there seems to be a common belief between teachers that students forget quickly (from one year to the next) and they have to be 'taught again'. The analysis shows that this did not happen to so great an extent with the experimental students, and most of students who were taught to concentrate on the *effect* of actions gained the concept and retained it into the next school year.

Not all students in the experimental group reached the higher stages of. Some may possibly benefit from more experience of concentrating on the *effect* of actions before they gain the benefit of such an exercise.

It is notable that, despite the distinction made in chapter between the triangle law and the parallelogram law, where the first was seen as more natural for combining journeys and the latter for combining forces at a point, in all three tests, only *one* student used the parallelogram law of addition. The use of triangles dominated the graphical mode and the symbolic mode deals with components individually in a way that also does not involve the parallelogram law. Therefore, apart from noting that the parallelogram law was rarely used, no comparison between the use of the two laws was possible from the written responses.

It was also evident from the post-test and the delayed post-test that the students from group A sketched with more understanding of equivalent vectors having the same direction and magnitude and were less likely to have the misconception that the addition of three given vectors required the vectors to be in the form a triangle. In the post-test and the delayed post-test, the main difference between the sketches of groups A and B is that many students in group A, especially in questions with different physical contexts, moved vectors around as 'free vectors', meaning that they treat the questions from a mathematical point of view, while that type of response was rare in group B.

Further triangulation is required in the form of comments of the teachers to gain insight into their views of how the students may perform, and, more particularly, into how students talk about their work. This triangulation will be performed in the next two chapters. In particular, the interviews with the students will be framed to give insight into how the students talk about the concepts and whether the more successful do have a different way of thinking of the concept of vector as a cognitive unit—a single entity with different uses in different contexts—or as a number of different concepts (force, journey, etc) which have very different properties.

The evidence of the use of vectors in different contexts already shows that the experimental students are likely to have a more coherent overall view of the notion of vector that can be applied in different contexts. The evidence of the handling of singular examples shows a greater degree of flexibility in using the notion of free vector. The greater use of the commutative law (which works for free vectors, but not for journeys) also shows that they are more likely to be operating fluently with free vectors.

In almost all respects (particularly in the use of the graphical mode), the quantitative evidence supports the three hypotheses, 1, 2 and 3, which together give quantitative support for the main hypothesis stated at the beginning of the chapter that:

> Teachers can help students develop a notion of a translation as free vector through building on physical experiences, leading to graphic and symbolic representations, with the notion of free vector being constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.