

**Effect as a pivot between actions and symbols:
the case of vector**

by

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CONTENTS

ACKNOWLEDGEMENT	ix
DECLARATION	x
ABSTRACT.....	xi
CHAPTER 1: THESIS OVERVIEW	1
<i>PROLOGUE</i>	1
1.1 INTRODUCTION.....	2
1.2 THE BACKGROUND TO THE RESEARCH	5
1.3 THE STRUCTURE OF THE THESIS	8
CHAPTER 2: LITERATURE REVIEW	11
2.1 INTRODUCTION.....	11
2.2 THEORIES OF KNOWLEDGE AND UNDERSTANDING	11
2.2.1 <i>Intuition</i>	11
2.2.2 <i>Instrumental-relational understanding</i>	14
2.2.3 <i>Procedural-conceptual knowledge</i>	15
2.3 DIFFERENT MODES OF OPERATION IN MATHEMATICS.....	18
2.3.1 <i>Successive stages of cognitive development</i>	19
2.3.2 <i>Construction of meaning</i>	20
2.3.3 <i>An example: the case of fractions</i>	22
2.3.4 <i>Embodiment of mathematical concepts in the physical world</i>	23
2.3.5 <i>The transition from embodiment to symbolism</i>	26
2.4 CONCEPT IMAGES AND COMPRESSION OF KNOWLEDGE.....	28
2.4.1 <i>Cognitive Units</i>	28
2.4.2 <i>Process-object encapsulation</i>	29
2.4.3 <i>Reflection</i>	31
2.5 BRINGING THEORIES TOGETHER	33
2.5.1 <i>Combining modes</i>	35
2.5.2 <i>Versatile thinking</i>	35
2.6 SUMMARY	36
CHAPTER 3: TOWARDS THE THEORETICAL FRAMEWORK.....	38
3.1 INTRODUCTION.....	38
3.2 THE SCHOOL BASED SITUATION	38
3.3 TEXT BOOKS ANALYSES	39
3.3.1 <i>How and when are vectors introduced in Physics?</i>	39
3.3.2 <i>How and when are vectors introduced in Mathematics?</i>	42
3.3.3 <i>Linking the text-book sequence to process-object theory</i>	48
3.4 RELEVANT EXAMPLES OF RESEARCH INTO MECHANICS.....	49
3.4.1 <i>Three vector concepts</i>	49
3.4.2 <i>Understanding of Mechanics</i>	51
3.4.3 <i>Understanding of force</i>	51
3.5 SUMMARY OF EVIDENCE AND FORMULATION OF A RESEARCH FRAMEWORK	53
3.5.1 <i>Theoretical framework perspective</i>	56
3.5.2 <i>The idea of ‘effect’</i>	57

Contents

CHAPTER 4: PRELIMINARY INVESTIGATIONS	67
4.1 INTRODUCTION	67
4.2 PRELIMINARY EMPIRICAL INVESTIGATION	68
4.3 SUMMARY TO PRELIMINARY EMPIRICAL INVESTIGATION	72
4.4 RELATING EMPIRICAL EVIDENCE TO THEORETICAL FRAMEWORK	74
CHAPTER 5: METHODS AND METHODOLOGY	77
5.1 INTRODUCTION	77
5.1.1 Method	77
5.1.2 Methodology	78
5.2 RESEARCH DESIGN	82
5.2.1 Sample	82
5.2.2 Triangulation	82
5.2.3 Variables	83
5.2.3.1 Prior variables	85
5.2.3.2 Intervening variables	85
5.2.3.3 Dependent variables	86
5.2.3.4 Consequent variables	86
5.2.4 <i>Qualitative and quantitative data collection instruments</i>	86
5.2.4.1 Questionnaire	86
5.2.4.2 Interviews	87
5.2.4.3 Lesson observation	88
5.3 TEACHING EXPERIMENT AND PLENARIES	88
5.4 SUMMARY	92
CHAPTER 6: PILOT STUDY	94
6.1 INTRODUCTION	94
6.2 DESIGN OF THE QUESTIONS	94
6.2.1 <i>Cognitive development of vector</i>	94
6.2.2 <i>Cognitive development of vector addition</i>	96
6.3 METHOD OF COLLECTING QUANTITATIVE DATA	100
6.3.1 <i>Quantitative Data Analysis of Understanding the Symbol of a Vector</i>	103
6.3.2 <i>Quantitative Data Analysis of Understanding Vector Addition</i>	112
6.4 METHOD OF COLLECTING QUALITATIVE DATA	120
6.5 QUANTITATIVE ANALYSIS OF THE RESULTS	121
6.6 CONCLUSIONS	124
CHAPTER 7: MAIN STUDY: QUANTITATIVE DATA ANALYSIS	126
7.1 INTRODUCTION	126
7.2 QUANTITATIVE DATA ANALYSIS OF UNDERSTANDING THE CONCEPT OF VECTOR & VECTOR ADDITION	127
7.2.1 <i>The General Case: Understanding the Symbol of Vector</i>	128
7.2.2 <i>The General Case: Understanding Vector Addition</i>	131
7.2.3 <i>Singular Cases: Understanding Vector Addition</i>	124
7.2.4 <i>Different contexts: Understanding Vector Addition</i>	138
7.2.5 <i>The commutative law in vector addition</i>	143
7.3 SUMMARY OF THE RESULTS	144
7.4 SUMMARY	149

Contents

CHAPTER 8: MAIN STUDY: QUALITATIVE DATA ANALYSIS	
INTERVIEWS WITH THE TEACHERS	152
8.1 INTRODUCTION.....	152
8.2 INTERVIEW WITH THE TEACHERS.....	152
8.3 GENERAL SUMMARY	164
CHAPTER 9 MAIN STUDY: QUALITATIVE DATA ANALYSIS	
INTERVIEWS WITH THE STUDENTS	167
9.1 INTRODUCTION.....	167
9.2 THE INTERVIEWS FOLLOWING THE PRE-TEST.....	168
9.2.1 Student S1.....	168
9.2.2 Student S2	174
9.2.3 Student S3.....	178
9.2.4 Student S4.....	181
9.3 THE INTERVIEWS FOLLOWING THE POST-TEST.....	185
9.3.1 Student S5.....	186
9.3.2 Student S6.....	189
9.3.3 Student S7.....	192
9.3.4 Student S8.....	196
9.3.5 Student S9.....	200
9.4 SUMMARY FROM THE INTERVIEWS	203
9.5 OVERALL TRIANGULATION BETWEEN THE INTERVIEWS AND THE QUANTITATIVE DATA.....	206
CHAPTER 10: SUMMARY AND PLANS FOR THE FUTURE RESEARCH.....	208
10.1 INTRODUCTION	208
10.2 THEORETICAL FRAMEWORK.....	211
10.3 THEMES OF THE TESTING.....	212
10.4 TESTING HYPOTHESIS.....	213
10.5 SUMMARY OF TESTING THEORY.....	217
10.6 LIMITATION OF THE STUDY	219
10.7 DIRECTIONS FOR FUTURE RESEARCH	219
10.8 REFLECTING ON THE EFFECT OF THE STUDY.....	221
<i>Epilogue.....</i>	<i>222</i>
REFERENCES.....	225
APPENDIX.....	230

Figures

<i>Fig. 1.1 The Triangle Law</i>	6
<i>Fig. 1.2. The Parallelogram Law</i>	6
<i>Fig. 1.3 compression from action to concept by focusing on the effect</i>	8
<i>Fig. 2.1 Example of the ‘intuitive’ addition of two vectors</i>	12
<i>Fig. 2.2 The first representation of $3/8$</i>	23
<i>Fig. 2.3 The second representation of $3/8$</i>	24
<i>Fig. 2.4 representation of equivalent fractions</i>	24
<i>Fig. 3.1 Forces in a horizontal direction</i>	40
<i>Fig. 3.2 Forces in a vertical direction</i>	40
<i>Fig. 3.3 Forces in several directions</i>	41
<i>Fig. 3.4 Resolving a force into horizontal and vertical directions</i>	41
<i>Fig. 3.5 A translation as a column vector</i>	43
<i>Fig. 3.6 A translation as an arrow from one point to another</i>	43
<i>Fig. 3.7 Translations as arrows with magnitude and direction</i>	43
<i>Fig. 3.8 Equivalent vectors and the special concept of position vector</i>	44
<i>Fig. 3.9 Equivalent vectors representing the same translation</i>	45
<i>Fig. 3.10 The triangle method of addition</i>	46
<i>Fig. 3.11 The parallelogram method of addition</i>	46
<i>Fig. 3.12 Position vectors in geometry</i>	46
<i>Fig. 3.13 Vector representations of geometrical positions</i>	47
<i>Fig. 3.14 Position vectors in geometrical figures</i>	47
<i>Fig. 3.15 Position vectors in terms of i and j</i>	47
<i>Fig. 3.16 Focusing on effect</i>	57
<i>Fig. 3.17 Cognitive development of vector</i>	59
<i>Fig. 3.18 Cognitive development of vector addition</i>	60
<i>Fig. 4.1 Two questions on forces (a slope)</i>	68
<i>Fig. 4.2 Preliminary Investigations questions</i>	69
<i>Fig. 4.3 Testing the visual sum of two vectors</i>	70
<i>Fig. 4.4 Embodied action</i>	72
<i>Fig. 5.1 Cognitive compression of the vector concept</i>	89
<i>Fig. 5.2 Action of translation—experimental lesson</i>	90
<i>Fig. 5.3 Action with two translations—experimental lesson</i>	90
<i>Fig. 5.4 Overall translation—experimental lesson</i>	90
<i>Fig. 5.5 Action of addition of two translations—experimental lesson</i>	91
<i>Fig. 5.6 Moving object—experimental lesson</i>	92
<i>Fig. 6.1 Test question 1</i>	94
<i>Fig. 6.2 Test question 2: adding vectors as arrows</i>	96
<i>Fig. 6.3 Test question 3: adding vectors in another way</i>	97
<i>Fig. 6.4 Test question 4: Add three vectors</i>	97
<i>Fig. 6.5 Test question 7: adding vectors in a drawing</i>	98
<i>Fig. 6.6 Test question 8: more sophisticated addition in a drawing</i>	99
<i>Fig. 6.7 Four stages of cognitive development of vector in the graphical mode</i>	103
<i>Fig. 6.8 Four stages of cognitive development of vector in the symbolic mode</i>	103
<i>Fig. 6.9 Test question 1</i>	104
<i>Fig. 6.10 Examples of students’ responses to test question 1</i>	105

Contents

Fig. 6.11 Questions 2(b) and 3(b).....	107
Fig. 6.12 Examples of four students' responses to questions 2(b), 3(b).....	107
Fig. 6.13 Singular question	109
Fig. 6.14 Allocation of stages to the responses to the singular case in 2(c), 3(c)	109
Fig. 6.15 Singular question	110
Fig. 6.16 Examples of students' responses to singular questions.....	111
Fig. 6.17 Stages of cognitive development of vector addition in the graphical mode.....	112
Fig. 6.18 Stages of cognitive development of vector addition in the symbolic mode	112
Fig. 6.19 Test question 2.....	113
Fig. 6.20 Test question 3.....	114
Fig. 6.21 Example of graphical responses to questions 2 (a) and 3 (a).....	114
Fig. 6.22 Examples of graphical responses to questions 2(a) and 3 (a)	115
Fig. 6.23 Example of graphical responses to questions 2 (b) and 3 (b).....	115
Fig. 6.24 Examples of symbolic responses to questions 2(b) and 3 (b).....	116
Fig. 6.25 Example of graphical responses to questions 2 (c) and 3 (c)	116
Fig. 6.26 Examples of graphical responses to questions 2(c) and 3 (c).....	117
Fig. 6.27 Test question 4: Add three vectors.....	117
Fig. 6.28 Response to question 4: Add three vectors.....	118
Fig. 6.29 Examples of responses to question set in the context of forces.....	119
Fig. 6.30 Examples of responses to question set in the context of displacements.	120
Fig. 6.31 Results of pilot pre-test.	122
Fig. 6.32 Results of pilot post-test.....	124
Fig. 7.1 Scatter graph of responses to all pre-test questions (vector).....	130
Fig. 7.2 Scatter graph of responses to all post-test questions (vector)	130
Fig. 7.3 Scatter graph of responses to all delayed post-test questions (vector)	130
Fig. 7.4 Scatter graph of responses to all pre-test questions on addition	133
Fig. 7.5 Scatter graph of responses to all post-test questions on addition.....	133
Fig. 7.6 Scatter graph of responses to all delayed post-test questions on addition.....	134
Fig. 7.7 Scatter graphs of responses to singular pre-test questions	136
Fig. 7.8 Scatter graphs of responses to singular post-test questions.....	137
Fig. 7.9 Scatter graphs of responses to singular pre-test questions	137
Fig. 7.10 Scatter graph of responses in the context of forces, pre-test.....	140
Fig. 7.11 Scatter graph of responses in the context of forces, post-test	140
Fig. 7.12 Scatter graph of responses in the context of forces, delayed post-test	140
Fig. 7.13 Scatter graph of responses in the context of displacements, pre-test	142
Fig. 7.14 Scatter graph of responses in the context of displacements, post-test.....	142
Fig. 7.15 Scatter graphs of response in the context of displacements, delayed post-test.....	142
Fig. 7.16 Comparative General Developments of Groups A and B	145
Fig. 7.17 Development of students through combined categories	146
Fig.7.18 Comparative development of Groups A and B (singular cases)	147
Fig. 7.19 Development of students through combined categories (singular cases)	147
Fig.7.20 Comparative developments of Groups A and B (different contexts).....	148
Fig. 7.21 Development of students through combined categories (context of forces).....	149
Fig. 8.1 Test question 1: Represent translation	152
Fig. 8.2 Test questions 2 and 3: Add two vectors	154
Fig. 8.3 Test question 4: Add three vectors	158
Fig. 8.4 Questions set in two different contexts	159
Fig. 8.5 Question on vector addition set in the diagram.....	162

Contents

Fig. 8.6 Singular question on vector addition set in the diagram.....	162
Fig. 9.1 Student S1: examples of responses to the pre-test	168
Fig. 9.2 Student S1: examples of responses to the pre-test	170
Fig. 9.3 Student S1: example of response to the pre-test.....	171
Fig. 9.4 Student S2: examples of responses to the pre-test	174
Fig. 9.5 Student S2: response to the interview question.....	175
Fig. 9.6 Student S2: the interview response to the singular question.....	175
Fig. 9.7 Student S2: responses to the different contexts pre-test questions.....	176
Fig. 9.8 Student S3: examples of responses to the pre-test	178
Fig. 9.9 Student S3: responses to different contexts questions in the pre-test	179
Fig. 9.10 Student S4: examples of responses to the pre-test	181
Fig. 9.11 Student S4: responses to different contexts questions in the pre-test	182
Fig. 9.12 Student S4: response to the interview question.....	183
Fig. 9.13 Student S5: responses to questions 2 & 3 in the post-test	186
Fig. 9.14 Student S5: corrected responses to question 2 during interview	187
Fig. 9.15 Student S5: responses to different contexts questions in the post-test.....	187
Fig. 9.16 Student S6: post-test responses to questions 2 & 3.....	190
Fig. 9.17 Student S6: post-test responses to the different contexts questions	191
Fig. 9.18 Student S7: responses to the pre-test questions 2 & 3.....	193
Fig. 9.19 Student S7: responses to the questions set in different contexts	194
Fig. 9.20 Student S8: post-test responses to questions 2 & 3.....	197
Fig. 9.21 Student S8: responses to the post-test questions set in different contexts	198
Fig. 9.22 Student S9: responses to the post-test questions 2 and 3	201
Fig. 9.23 Student S9: responses to the post-test questions set in different contexts	202
Fig.10.1 Question on forces (a slope).....	222
Fig. 10.2 Revisiting the original problem.....	223

Tables

<i>Table 2.1 The fundamental cycle of conceptual construction</i>	<i>34</i>
<i>Table 3.1 Development of the vector concept, combining graphic and symbolic.....</i>	<i>64</i>
<i>Table 3.2 Development of vector addition, combining graphic and symbolic.....</i>	<i>65</i>
<i>Table 4.1 Effect of embodied approach in reflective plenaries.....</i>	<i>72</i>
<i>Table 6.1 Table of the second stage of the categorisation</i>	<i>102</i>
<i>Table 7.1 Graphical responses to test questions.....</i>	<i>128</i>
<i>Table 7.2 Symbolic responses to test questions.....</i>	<i>129</i>
<i>Table 7.3 Graphical responses to the test questions.....</i>	<i>132</i>
<i>Table 7.4 Symbolic responses to the test questions.....</i>	<i>132</i>
<i>Table 7.5 Graphical responses to the singular questions.....</i>	<i>135</i>
<i>Table 7.6 Symbolic responses to the singular questions.</i>	<i>135</i>
<i>Table 7.7 Graphical responses to questions set in different contexts</i>	<i>138</i>
<i>Table 7.8 Symbolic responses to questions set in different contexts.....</i>	<i>139</i>
<i>Table 7.9 Responses using the commutative law of addition</i>	<i>143</i>

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Declaration

I hereby certify that this thesis is my own work. Portions of the Preliminary Study chapter (Chapter 4) were used in two papers: one submitted to the 26th Annual Conference of the International Group for the Psychology of Mathematics Education (PME) in Norwich, UK, in 2002; a second one submitted to the Mediterranean Journal of Mathematical Education in Cyprus, in 2003. The references to the papers are included in the References of this thesis. I also confirm that this thesis has not been submitted for a degree at any other university.

Abstract

Students' difficulties with vectors in Mechanics, at AS and A level, have been considered in a number of studies to date. Some of the research considers how students' intuitions arise from working in different contexts and how it affects their problem-solving capabilities, others think that pre-Newtonian views affect students' thinking. There are considerations that the idea of vector has different meanings in different contexts and therefore it is not easily transferable from one context to another. There are suggestions in the literature that a qualitative approach to teaching would help students to learn. None of the research studied considers the important idea of focusing on the vector concepts that are *common* to the various contexts, instead they are more concerned with the problems caused by the *differences* between them. Nor do they focus on the compression of a vector as an action into the more flexible idea of a free vector as a single mental object that can be represented by any arrow of given magnitude and direction.

In this thesis an approach is developed to base the students' experience on manipulating physical objects, to focus on the *effect* of a translation rather than on the action itself. The essential idea is to notice that every point on the object translated (and on the hand doing the translation) moves in the same direction and travels the same distance. The *effect* of the translation is therefore represented by *any* arrow of this magnitude and direction, leading to the notion of free vector. From the same viewpoint, the sum of two vectors is simply the single translation having the same effect as the combination of one translation followed by the other.

The main hypothesis is that:

Teachers can help students develop the notion of a translation as a free vector through focusing on the effects of physical actions, linking graphic and symbolic representations, so that the concept of free vector is constructed as a cognitive unit that may be used in a versatile way in a range of different contexts.

This was tested by a comparative study of two classes using both quantitative and qualitative methods. The control group carried on with the normal programme of study while the experimental group was exposed to lessons focusing on the notion of a free vector as the effect of a transformation. The students' own constructions were supported by activities and discussions in reflective plenary sessions. The results of the study revealed that there were significant changes in the students following the experimental programme, in which they were more likely to conceive of the symbols for vectors as cognitive units operating in a flexible and versatile manner. The quantitative improvement was sustained and increased over a longer period.

Interviews with the teachers revealed differences between the mathematics and physics teachers' perceptions of the students' expected difficulties. Interviews with the students revealed the more successful interviewees referring to the concept of vector as a single cognitive unit across contexts, while the less successful tended to consider the concept of vector operating in different ways as journeys and as forces.

Both quantitative and qualitative data show significant conceptual changes in students following the experimental approach; these changes were more marked over the longer period of time between pre-test and delayed post-test.